

Math Problem

a- If A has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$ and corresponding eigenvectors $u_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$, $u_2 = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$

Solve the initial value problem $\dot{u}(t) = Au(t)$ with $u(0) = \begin{Bmatrix} 6 \\ 3 \end{Bmatrix}$.

b- True or false and why

1. if A is diagonalizable, then A has no multiple eigenvalues.
2. if λ^2 is an eigenvalue of A^2 , then λ is an eigenvalue of A .
3. if A is an $n \times n$ real matrix having real eigenvalues and n orthogonal eigenvectors, then A is symmetric.
4. every invertible matrix is diagonalizable.
5. if $A = xy^T$, where $x, y \in \mathbb{R}^n$ then $\lambda = x^T y$ is an eigenvalue of A .
6. if $A = pBp^T$ with p invertible, then A and B have the same eigenvalues.

Math Problem

Consider the linear general second-order PDE:

a) $A(x, y)U_{,xx} + B(x, y)U_{,xy} + C(x, y)U_{,yy} + DU_{,x} + EU_{,y} + FU = G(x, y)$

Give the three classifications for this PDE, i.e., elliptic, ...

b) What kind of PDE is the following

1) Heat equation $C \frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial t} = 0$

2) Wave equation $C^2 \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial t^2} = 0$

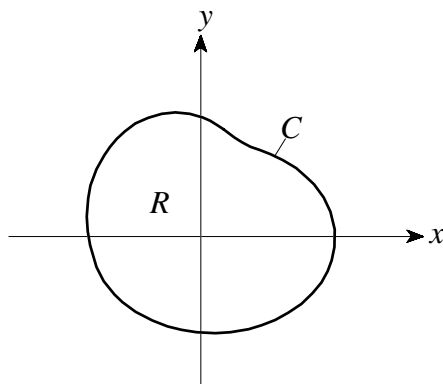
3) Laplace equation $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$

c) Solve the following equation:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 2e^{5t} \\ 3e^{2t} \end{Bmatrix}$$

Math Problem

Let R be a closed bounded region in the x - y plane with the curved boundary C .



Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\partial F_1 / \partial y$ and $\partial F_2 / \partial x$ everywhere in some domain containing R . Then, Green's theorem states:

$$\int_A \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \oint_C (F_1 dx + F_2 dy). \quad (1)$$

(a) Use (1) to show that the area of R , A is

$$A = \frac{1}{2} \oint_C (x dy - y dx). \quad (2)$$

(b) Use (2) to calculate the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (3)$$

Math Problem

a) If $[A]$ is singular, what can you say about the following:

- 1) $\det[A]$
- 2) eigenvalues of $[A]$
- 3) relationship between the rows and columns of $[A]$
- 4) the solutions of the inhomogeneous set of linear equations, $[A]\{x\} = \{F\}$
- 5) the solutions of the homogeneous set of linear equations, $[A]\{x\} = \{0\}$

b) Solve the following:

$$\frac{dx}{dt} = -x + 2y$$

$$\frac{dy}{dt} = -2x - y$$

Math Problem

If A has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$ and corresponding eigenvectors

$$u_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}, \quad u_2 = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$$

a) Solve the initial value problem $\dot{u}(t) = Au(t)$ with $u(0) = \begin{Bmatrix} 6 \\ 3 \end{Bmatrix}$.

b) True or false and why:

1. if A is diagonalizable, then A has no multiple eigenvalues.
2. if λ^2 is an eigenvalue of A^2 , then λ is an eigenvalue of A .
3. if A is an $n \times n$ real matrix having real eigenvalues and n orthogonal eigenvectors, then A is symmetric.
4. every invertible matrix is diagonalizable.
5. if $A = xy^T$, where $x, y \in \mathbb{R}^n$ then $\lambda = x^T y$ is an eigenvalue of A .
6. if $A = pBp^T$ with p invertible, then A and B have the same eigenvalues.

Math Problem

For the initial/boundary value problem

$$\text{DE:} \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L, \quad 0 \leq t$$

$$\text{BC's:} \quad u(0,t) = 0 \quad \text{and} \quad \frac{\partial u(L,t)}{\partial x} = 0 \quad \text{for} \quad 0 \leq t$$

$$\text{IC:} \quad u(x,0) = f(x) \quad \text{for} \quad 0 \leq x \leq L$$

carry out the following steps:

- a) Formulate the eigenvalue problem obtained through separation of variables
- b) Find the characteristic equation whose roots are the eigenvalues
- c) Find the eigenvalues and eigenfunctions and show that the eigenfunctions are orthogonal
- d) Write out the eigenfunction expansion of the solution $u(x,t)$
- e) Evaluate the constants in part (d) for the special case $f(x) = 1$

Math Problem

(a) Given $x^2/16 + y^2/12 + z^2/9 = 1$ and z is a dependent variable while x and y are independent, find $\partial z / \partial x$.

(b) Find the following:

$$\lim_{x \rightarrow 0} \frac{(e^x - 1)\sin x}{\cos x - \cos^2 x}.$$

(c) Find the point of the plane $2x - 3y - 4z = 25$ which is the nearest to the point $(3,2,1)$.

Elasticity Problem

The stress tensor in the principal direction is given by

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}.$$

Show that the shearing traction S on an infinitesimal area with normal (n_1, n_2, n_3) is given by

$$S^2 = (\sigma_1 - \sigma_2)^2 n_1^2 n_2^2 + (\sigma_1 - \sigma_3)^2 n_1^2 n_3^2 + (\sigma_2 - \sigma_3)^2 n_2^2 n_3^2 .$$

Elasticity Problem

Let $I(t)$ be a volume integral of a continuously differentiable function $A(x_i, t)$ defined over a spatial domain $V(x_i, t)$ occupied by a given set of material particles, that is

$$I(t) = \int_V A(x_i, t) dV \quad (1)$$

- (a) Write down the volume expression for the material derivative of I , namely $\frac{DI(t)}{Dt}$.
- (b) Write in index notation the statement of conservation of angular momentum over the volume element $V(x_i, t)$.
- (c) Use the results of (a) to show that the conservation of the angular momentum in (b) reduces to showing that the stress tensor is symmetric, namely, $\sigma_{ij} = \sigma_{ji}$.

Elasticity Problem

Let us consider a circular cylinder of length L and radius a oriented so that its axis coincides with the x_1 axis of a Cartesian coordinate system $x_1 x_2 x_3$. The material is isotropic and there are no body forces present. The state of stress is given by the following stress tensor

$$[\sigma] = \begin{bmatrix} Ax_2 + Bx_3 & Cx_3 & -Cx_2 \\ Cx_3 & 0 & 0 \\ -Cx_2 & 0 & 0 \end{bmatrix}.$$

- a) Show that the equilibrium equations are satisfied.
- b) Show that the compatibility conditions are also satisfied.
- c) Show that free boundary conditions prevail on the surface of the cylinder.

Elasticity Problem

$$\sigma_{ij} = \begin{bmatrix} x^2yz & xy & yz \\ xy & xy^2z & x^2yz^2 \\ yz & x^2yz^2 & xyz^2 \end{bmatrix}$$

For the stress tensor given above for a body with no acceleration and no couple stresses you are to:

- a) At the point (1,2,3), determine principal stresses and the direction of one of the principal stresses.
- b) At the same point as above, find the octahedral normal and octahedral shear stresses.
- c) For a linearly elastic material, find the strains in terms of (x,y,z) , that are associated with the given stress tensor.
- d) Find the change in the angle between the lines that go through the points (1,2,3) and (2,4,5) and the points (1,2,3) and (7,4,1) for the above strain state.
- e) Show that an infinitesimal displacement du_j consists of an infinitesimal strain and rotation component.

Elasticity Problem

Consider the principal stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

a) Show that

$$\left(\frac{\mu}{T_1} \right)^2 + \left(\frac{\mu}{T_2} \right)^2 + \left(\frac{\mu}{T_3} \right)^2 = 1$$

b) Show that

$$\mu_1^2 = \frac{(\sigma_N - \sigma_2)(\sigma_N - \sigma_3) + S^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)},$$

where σ_N is the normal component of the stress vector along $\vec{\mu}$ and S is the shear stress.

Elasticity Problem

Show for plane-strain problems that the general equations of linear isotropic elasticity lead to the biharmonic equation

$$\nabla^4 \Phi + \frac{1-2\nu}{1-\nu} \nabla^2 V = 0,$$

where Φ is the Airy stress potential defined by $\tau_{xx} = \frac{\partial^2 \Phi}{\partial y^2}$, $\tau_{yy} = \frac{\partial^2 \Phi}{\partial x^2}$, and $\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$, and

V is the body force potential such that $F_x = -\frac{\partial V}{\partial x}$ and $F_y = -\frac{\partial V}{\partial y}$.

Elasticity Problem

Consider a continuum region having the volume V and surface area S .

(a) If
$$I(t) = \int_V A(x_i, t) dV$$

Show details of the derivation leading to

$$\frac{DI}{Dt} = \int_V \left(\frac{DA}{Dt} + A \frac{\partial v_i}{\partial x_i} \right) dV,$$

where v_i is the velocity vector and $\frac{DI}{Dt}$ is the total time derivative of the function $I(t)$.

(b) Give the statement of conservation of linear momentum and show, using the results of (a), that

$$\rho \frac{Dv_i}{Dt} = \sigma_{ij,j} + X_i,$$

where X_i is the body force vector and σ_{ij} are the stress tensor components.

Elasticity Problem

Consider an elastic isotropic solid. Use the equations of equilibrium and the constitutive and compatibility relations to show that

$$\sigma_{ij,kk} + \frac{1}{1+\nu} \sigma_{kk,ij} + (F_{i,j} + F_{j,i}) + \frac{\nu}{1-\nu} F_{k,k} \delta_{ij} = 0$$

Elasticity Problem

Consider a solid body subjected to the stress σ_{ij} .

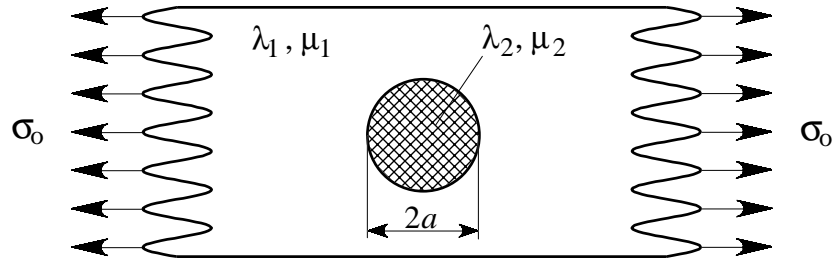
- (a) Using the index notation, determine the expression for the shear stress τ on the cut whose normal is \bar{n} .
- (b) Specialize this shear stress to the case where the components of the shear stress σ_{ij} vanish and show that τ^2 reduces to

$$\tau^2 = n_1^2 n_2^2 (\sigma_{11} - \sigma_{22})^2 + n_1^2 n_3^2 (\sigma_{11} - \sigma_{33})^2 + n_2^2 n_3^2 (\sigma_{22} - \sigma_{33})^2$$

- (c) Use the results of (b) to calculate the octahedral shear stress.

Elasticity Problem

An unbounded elastic plate contains a circular inclusion of a different elastic material. The plate is loaded in tension as shown in the figure. Determine the stress distribution in both the plate and the inclusion.



Elasticity Problem

Use the equations of Equilibrium

$$\sigma_{ij,j} + F_i = 0$$

with the constitutive relation

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij}$$

and compatibility

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik}$$

to show that

$$\sigma_{ij,kk} + \frac{1}{1+\nu} \sigma_{kk,ij} + (F_{i,j} + F_{j,i}) + \frac{\nu}{1-\nu} F_{k,k} \delta_{ij} = 0.$$

If needed, use the relation

$$\nu = \frac{\lambda}{2(\lambda + \mu)}.$$

Elasticity Problem

Use the Field Equations

$$\sigma_{ij,j} + F_i = 0 \quad (1)$$

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} \quad (2)$$

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \quad (3)$$

and show that

$$\sigma_{ij,kk} + \frac{2(\lambda + \mu)}{3\lambda + 2\mu} \sigma_{kk,ij} + F_{i,j} + F_{j,i} + \frac{\lambda}{\lambda + 2\mu} F_{k,k} \delta_{ij} = 0. \quad (4)$$

Hints:

- (i) Start by specializing equation (3) to $k = l$ and call it equation (5).
- (ii) Also specialize equation (3) to $i = k$ and $j = l$ and call it equation (6).
- (iii) Use equations (1), (2), (5) and (6) to show (4).

Strength of Materials Problem

Let us consider a rotating flat disk of inner and outer radii b and a , respectively. In terms of the radial displacement u , the radial equilibrium equation can be written as follows:

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (ru) \right] = -\frac{1-\nu^2}{E} \rho \omega^2 r.$$

Solve this differential equation and derive equations for the radial $\sigma_r(r)$ and hoop $\sigma_\theta(r)$ stress distributions by assuming plane stress conditions. What is the ratio of the maximum hoop stress to the maximum radial stress with no boundary loading? What should be the $\xi = a/b$ ratio to assure that $\sigma_{\theta \max} / \sigma_{r \max} = 4$ if the Poisson ratio is $\nu = 1/3$.

Strength of Materials Problem

A very long, prismatic elastic shaft of elliptical cross section is embedded in an elastic medium. The shear modulus G of the shaft and the two semi-axes a and b of the elliptical cross section are known. When a length dx of the shaft rotates an amount θ , the medium applies a retaining torque $dT = k\theta dx$ to the length dx . Let us assume that a torque T_0 is applied to the end of the shaft at $x = 0$. Obtain an expression for the rotation angle $\theta(x)$.

Strength of Materials Problem

Let us consider the torsional deformation of a prismatic bar of elliptical cross section. Calculate the ratio between the maximum strain energy density $U_{0\max}$ and the average strain energy density $U_{0\text{avg}}$ as a function of the aspect ratio $c = a / b$ of the cross section (a and b are the longer and shorter semi-axes of the cross section, respectively). How does this strain energy density ratio $U_{0\max} / U_{0\text{avg}}$ compare to the same ratio for a prismatic bar of narrow rectangular cross section?

Strength of Materials Problem

Let us consider a homogeneous, isotropic, and linearly elastic disk under the influence of an axisymmetric temperature distribution $T = T(r)$. Assume that Young's modulus E , Poisson's ratio ν , and the thermal expansion coefficient α are known. Derive an equation for the distribution of the radial stress $\sigma_r(r)$. How can you specialize this result to the case of a disk whose inside and outside boundaries at $r = b$ and $r = a$, respectively, are free of tractions?

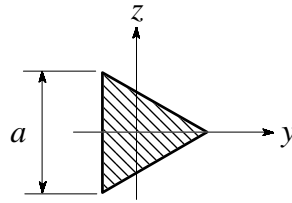
Strength of Materials Problem

Consider a thick-walled cylinder with outer and inner radii of a and b , respectively, subjected to an internal pressure of p . Poisson's ratio ν and Young's modulus E are known. Calculate the difference in the deformed wall thickness between plane-stress and plane-strain conditions.

Strength of Materials Problem

Consider a bicycle wheel of radius a rotating at angular velocity ω . Assume that all spokes are radial, lie in the same plane, and are of the same material as the rim. Let A_r and A_s be cross-sectional areas of the rim and a single spoke, respectively. Also assume that n , the number of spokes, is large and the stresses are zero when $\omega = 0$. Derive equations for the hoop stress σ_θ in the rim and the maximum radial stress σ_{rm} in the spokes. What is the asymptotic value of the $\sigma_{rm} / \sigma_\theta$ ratio for very small and very large $n A_s / A_r$ cross-sectional area ratios between the spokes and the rim?

Strength of Materials Problem



Apply the Saint-Venant torsion theory to a solid bar whose cross section is an equilateral triangle. Determine the Prandtl stress function ϕ ($\tau_{xy} = \partial\phi/\partial z$ and $\tau_{xz} = -\partial\phi/\partial y$) by satisfying the $d\phi = 0$ boundary condition at all three sides. Assume that for a given rate of twist β and shear modulus G the solution satisfies the $\nabla^2\phi = -2G\beta$ compatibility equation. Calculate the torsional rigidity $C = T/(\beta G)$, where $T = 2\iint\phi dA$ is the torque.

Strength of Materials Problem

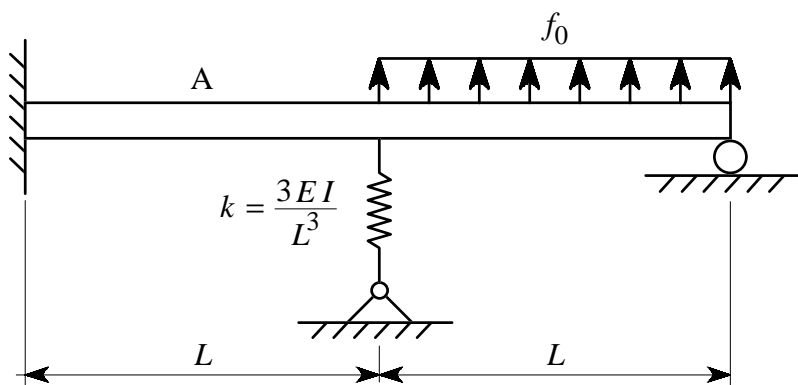
Consider a thick-walled cylinder with outer and inner radii of a and b , respectively, under the influence of outer and inner pressures p_o and p_i , respectively. Assume that plane stress conditions prevail, i.e., $\sigma_z = 0$ and that Poisson's ratio ν and Young's modulus E are known. Calculate the radial displacement u as a function of the radial coordinate r . What is the radial displacement $u(r)$ in the simple case of hydrostatic pressure, i.e., when $p_i = p_o$?

Strength of Materials Problem

Consider the torsional stiffness $k = T / \beta$ of a prismatic bar, where T is the applied torque and β is the resulting rate of twist. According to Saint-Venant's approximate formula for prismatic bars of arbitrary cross section, $k \approx GA^4 / (4\pi^2 J)$, where G , A , and J denote the shear modulus, cross sectional area, and polar moment, respectively. Determine how accurate this approximation is for a prismatic bar of elliptical cross section with an aspect ratio of 3.

Structural Mechanics Problem

For the beam configuration and loading condition shown below, you are to find all reactions and the vertical deflection at point A (located at the mid-point of the left side of the beam). Also draw the shear and bending moment diagrams, with all the important points clearly labeled and with their values shown. Assume that Young's modulus E and the moment of inertia I are known for the beam.



Structural Mechanics Problem

For the triply redundant frame shown below:

- (a) Find all reactions.
- (b) Draw the shear and bending moment diagrams for each member of the frame.
- (c) Find the deflection at point A located at the center of the horizontal bar.

