A LOCAL/GLOBAL SANDWICH HOMOGENIZATION PROCEDURE FOR FINITE ELEMENT SIMULATION

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ABSTRACT

A new finite element approach to sandwich shells is proposed. It uses existing shell finite elements formulated for homogeneous shells. The "sandwich" nature of the problem is hidden from the main finite element program. Based on several assumptions the proposed homogenization procedure calculates stress increments in a homogeneous fictitious material, called "equivalent", which correspond to the strain increments in the equivalent material points. The stresses in the equivalent material are calculated based on the stress and strain fields in the sandwich layers, which are determined from the incoming strain field for the equivalent material. This procedure is most suitable for including strain softening and other material nonlinear behavior. The approach can be combined with shell elements formulated for homogeneous materials, based on Reissner-Mindlin shell theory or with elements based on a higher order shell theory. It avoids the necessity of formulating special shell elements for sandwich constructions, which in most cases, due to their large number of degrees of freedom, significantly decrease the computational efficiency of the finite element analysis. The sandwich homogenization procedure is combined with a composite micromechanics-based model for woven composites to analyze sandwich shells with woven fabric faces.

NOMENCLATURE

In the Formulation of the Sandwich Homogenization Procedure

C_{ij} = elements of the tangential constitutive matrix
N_{ij} = resultant cross-sectional forces
M_{ij} = resultant cross-sectional moments
h_c = core thickness
h_f = face thickness
h = total shell thickness ($h = 2h_f + h_c$)
$d\varepsilon_{ij}$ = components of the incremental strain tensor
$d\varepsilon_{ij}^0$ = strain increments in the shell midplane
$d\kappa_{ij}$ = cross-section rotation increments

Symposium of Recent Advances in Mechanics of Aerospace Structures and Materials, AD-Vol 56, pp. 219-235, ASME winter meeting, Anaheim, CA, Nov. 1998.

$d\sigma_{ij}$ = components of the incremental stress tensor
$(\ )^b$ = referring to the bottom face
$(\ )^c$ = referring to the core
$(\ )^e$ = referring to the equivalent shell
$(\ )^f$ = referring to the faces
$(\ )^s$ = referring to the sandwich shell
$(\ )^t$ = referring to the top face

In the Formulation of the Composite Micromechanics Model

a = half length of the representative cell
b = half width of the representative cell
H = height of the representative cell
H_t = height of fiber tows
l_x = length of a subcell
l_y = width of a subcell
L_x = length of the representative cell
L_y = width of the representative cell
S_{ij} = compliance components in principal material system
ψ_{ij} = compliance components of constituents in global coordinate system
Ψ_{ij} = compliance components of the whole cell in global coordinate system
V = volume of a representative cell
θ_f = local angle between the fill yarn and global coordinate system
θ_w = local angle between the warp yarn and global coordinate system
$(\ )^{(\alpha,\beta)}$ = average quantities of a subcell
$\bar{(\ )}$ = average quantities of representative volume cell
$(\ )^t$ = quantities of tangential stiffness matrix or compliance matrix
$(\ )^*$ = in-plane or out-of-plane strains or stresses in principal material coord. system
$(\ )_i$ = in-plane stress or strain components
$(\ )_o$ = out-of-plane stress or strain components
$(\ )_f$ = quantities of the fill tow
$(\ )_w$ = quantities of the warp tow
$(\ )_m$ = quantities of the matrix

INTRODUCTION

Due to their numerous advantages sandwich shells are widely used in modern design of all kinds of structures: land, marine and space vehicles, civil and other structures. Many of their properties – geometrical, acoustic, thermal, etc. can be tailored to suit specific design objectives. One of the major advantages of sandwich shells is their very high weight-to-stiffness and weight-to-strength ratio compared to other commonly used structural materials. Along with that, being a combination of several materials with quite different physical properties makes the analysis of sandwich shells difficult and the results produced are often unreliable and far from real behavior. Therefore, approaching the analysis of sandwich shells from theoretical, experimental, computational point of view has been a major challenge for research within the recent decades. Much of the theoretical and computational research carried out in this field utilizes the finite element approach, which has long proved its efficiency and strength in solving problems of different types. Among the recent work in sandwich shell analysis are the studies of Frostig and Baruch (1994), Bozhevolnaya and Frostig (1997), Hause and Librescu (1998), Oskooei and Hansen (1998) just to mention a few. The major approaches and achievements in that area have been systemized in some review works [see Ha (1990), Noor et al. (1996)], where future trends and topics of interest are also outlined. To the best of our knowledge all present work involving finite element approach to

sandwich shell structures is done on new shell finite element formulation. That is, the different materials comprising the sandwich shell are combined in some way, based on Kirschhoff, Reissner-Mindlin, or higher order shell theory, producing a new finite element. Most of these can not or are hard to adapt to be able to analyze sandwich shells with composite faces. The present study utilizes a rather different approach – it develops a new homogenization procedure formulated for sandwich shells. Based on some assumptions the proposed homogenization procedure calculates stresses in a homogeneous fictitious material, called "equivalent", which correspond to the calculated strains in the equivalent material points. The stresses in the equivalent material are calculated based on the stress and strain fields in the sandwich layers, which are determined from the incoming strain field for the equivalent material, and are available for failure checks or other computations. This approach can be combined with shell elements formulated for homogeneous materials, based on Reissner-Mindlin shell theory or with elements based on a higher order shell theory. It avoids the necessity of formulating special shell elements for sandwich constructions. The sandwich homogenization procedure is combined with a composite micromechanics-based model for woven fabrics (Tabiei and Jiang, 1998) to analyze sandwich shells with woven fabric faces.

FORMULATION OF THE SANDWICH HOMOGENIZATION PROCEDURE

The proposed sandwich homogenization procedure performs the standard task of any finite element material model: receiving the strain increments in the equivalent material calculate the corresponding stress increments. For this homogenization procedure to work it is necessary to process all integration points throughout a given cross-section simultaneously. This is due to the fact that the sandwich nature of the problem is hidden from the main program and is only considered in the homogenization procedure. There, based on assumptions relating the equivalent homogeneous shell and the sandwich shell, from the strain increment distribution in the equivalent material cross-section the stress increments in the sandwich and in the equivalent material cross-section are calculated. The assumptions that the sandwich homogenization procedure is based upon are:

- *The strain increment distribution functions throughout the cross-section of the equivalent material are known*. These functions can be defined based on the element formulation: the displacement field within the element and the strain-displacement relations. However, since the sandwich homogenization procedure is based on forces, moments, and strain energy integral equalities, the strain increment distribution functions are not used but rather the values of integrals involving them are needed. These integrals can be easily calculated using the values of the strain increments in the integration points throughout the thickness and some numerical integration rule;
- *The strain increment distribution and the values of the strain increments throughout the thickness of the sandwich shell are the same as that of the equivalent shell*. Although this assumption violates some of the continuity conditions on the sandwich layers interfaces, it has a major contribution to the simplicity and computational efficiency of the homogenization procedure. However, if it is combined with a higher order shell element these inconsistencies will probably be insignificant. Or, if needed, the distribution for some of the strain increments in the sandwich cross-section can be altered to satisfy the continuity conditions;
- *The faces and core layers of the sandwich shell and the equivalent shell material are assumed homogeneous and orthotropic*. For composite sandwich faces this is obviously not true, but then a composite homogenization model can be used to produce a homogeneous face layer. The present study involves a micromechanics-based homogenization model for sandwich faces made of woven composite layers;
- *The "equivalence" between the fictitious equivalent shell and the sandwich shell is based on assumed equalities of the cross-sectional forces, moments, and the strain energy for the cross-section*;
- Although the proposed material model can be developed for an unsymmetrical sandwich shell, *in the present study a symmetric sandwich shell is assumed*. Both faces have the same thickness, h_f, and are of the same material. The thickness of the core is denoted with h_c.

The formulation of the proposed homogenization procedure is rather simple. It receives from the finite element main program the values of the strain increments for the integration points of the current cross-section. Knowing the shell formulation that the current shell element is based upon, the strain increment distribution functions throughout the shell cross-section can be determined. Assuming same distribution and same values for the strains in the sandwich shell cross-section and based on the known material constitutive relations, the stress distribution and corresponding values can be calculated. The resultant forces and moments for the cross-section

can also be calculated. Assuming that the stress increments in the equivalent material produce the same values for the resultant forces and moments, they can be expressed through the strain increment distribution functions and the unknown constitutive relation constants of the equivalent material. Assuming orthotropy for the equivalent material yields nine unknown constitutive relation constants, C_{ij}^{e} in Eq. (1.1), and eight equations Eq. (1.2) relating the corresponding forces and moments in the equivalent and the sandwich cross-section. The ninth equation Eq. (1.4), necessary to find the unknowns, can be acquired by assuming equality of the strain energy increment of both the equivalent and the sandwich sections. This results in a system of nine linear equations with nine unknowns, which can be solved and thus C_{ij}^{e} are determined. Then the corresponding stresses in the equivalent material are calculated. Note that the system of equations uncouples to a subsystem of six equations with six unknowns, C_{ij}^{e} for $i,j = 1,2,3$, and three separate equations with one unknown each for C_{44}^{e}, C_{55}^{e}, and C_{66}^{e}.

$$\begin{Bmatrix} d\sigma_{xx}^{e} \\ d\sigma_{yy}^{e} \\ d\sigma_{zz}^{e} \\ d\sigma_{yz}^{e} \\ d\sigma_{xz}^{e} \\ d\sigma_{xy}^{e} \end{Bmatrix} = \begin{bmatrix} C_{11}^{e} & C_{12}^{e} & C_{13}^{e} & 0 & 0 & 0 \\ C_{12}^{e} & C_{22}^{e} & C_{23}^{e} & 0 & 0 & 0 \\ C_{13}^{e} & C_{23}^{e} & C_{33}^{e} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^{e} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}^{e} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66}^{e} \end{bmatrix} \begin{Bmatrix} d\varepsilon_{xx} \\ d\varepsilon_{yy} \\ d\varepsilon_{zz} \\ d\varepsilon_{yz} \\ d\varepsilon_{xz} \\ d\varepsilon_{xy} \end{Bmatrix} \tag{1.1}$$

The forces and moments that are involved in the model are:

N_{xx}, N_{yy} – the resultant normal forces for cross-sections with normals x and y-axis respectively;
N_{zz} – the resultant force of $d\sigma_{zz}$;
N_{xz}, N_{yz} – the transverse shear forces in cross-sections with normals x and y-axis respectively;
$N_{xy} = N_{yx}$ – the shear force in the shell plane;
M_{yx}, M_{xy} – the moments of the normal stress increments $d\sigma_{yy}$ and $d\sigma_{xx}$ about x and y-axis respectively;

A Cartesian coordinate system, xyz, is fixed at the midplane between the upper and lower surfaces of the shell. The x and y axes lie in the shell plane, and the z axis is normal to the plane in the direction of the bottom face. Let h denote the total thickness of the equivalent and the sandwich shell. Then $h = 2h_f + h_c$. The equalities of the resultant forces and moments for the cross-section will yield:

$$N_{ij}^{e} = N_{ij}^{s} \rightarrow \int_{-\frac{h}{2}}^{\frac{h}{2}} d\sigma_{ij}^{e} dz = \int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{ij}^{t} dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\sigma_{ij}^{c} dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{ij}^{b} dz \ , \quad i, j = x,y,z$$

$$M_{xy}^{e} = M_{xy}^{s} \rightarrow \int_{-\frac{h}{2}}^{\frac{h}{2}} d\sigma_{xx}^{e} z dz = \int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{xx}^{t} z dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\sigma_{xx}^{c} z dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{xx}^{b} z dz \tag{1.2}$$

$$M_{yx}^{e} = M_{yx}^{s} \rightarrow \int_{-\frac{h}{2}}^{\frac{h}{2}} d\sigma_{yy}^{e} z dz = \int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{yy}^{t} z dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\sigma_{yy}^{c} z dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{yy}^{b} z dz$$

Here the superscripts e, s, t, c and b denote equivalent, sandwich, top face, core and bottom face respectively. Expressing the stress increments in the integrals through the strain increments from the constitutive equations, Eq. (1.1) yields:

$$C_{11}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx}dz + C_{12}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy}dz + C_{13}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz}dz = C_{11}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{xx}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{xx}dz\right) +$$

$$+ C_{12}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{yy}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{yy}dz\right) + C_{13}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{zz}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{zz}dz\right) + C_{11}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx}dz + C_{12}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy}dz + C_{13}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz}dz$$

$$C_{12}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx}dz + C_{22}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy}dz + C_{23}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz}dz = C_{12}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{xx}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{xx}dz\right) +$$

$$+ C_{22}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{yy}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{yy}dz\right) + C_{23}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{zz}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{zz}dz\right) + C_{12}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx}dz + C_{22}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy}dz + C_{23}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz}dz$$

$$C_{13}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx}dz + C_{23}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy}dz + C_{33}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz}dz = C_{13}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{xx}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{xx}dz\right) +$$

$$+ C_{23}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{yy}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{yy}dz\right) + C_{33}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{zz}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{zz}dz\right) + C_{13}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx}dz + C_{23}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy}dz + C_{33}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz}dz$$

$$C_{44}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yz}dz = C_{44}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{yz}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{yz}dz\right) + C_{44}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yz}dz$$

$$C_{55}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xz}dz = C_{55}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{xz}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{xz}dz\right) + C_{55}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xz}dz \tag{1.3}$$

$$C_{66}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xy}dz = C_{66}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{xy}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{xy}dz\right) + C_{66}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xy}dz$$

$$C_{11}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx}zdz + C_{12}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy}zdz + C_{13}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz}zdz = C_{11}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{xx}zdz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{xx}zdz\right) +$$

$$+ C_{12}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{yy}zdz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{yy}zdz\right) + C_{13}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{zz}zdz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{zz}zdz\right) + C_{11}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx}zdz + C_{12}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy}zdz + C_{13}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz}zdz$$

$$C_{12}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx} z dz + C_{22}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy} z dz + C_{23}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz} z dz = C_{12}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{xx} z dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{xx} z dz\right) +$$

$$+ C_{22}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{yy} z dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{yy} z dz\right) + C_{23}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{zz} z dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{zz} z dz\right) + C_{12}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx} z dz + C_{22}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy} z dz + C_{23}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz} z dz$$

Here the superscript *f* denoted top and bottom faces. The strain energy equation in terms of the stresses and strains is

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} d\sigma_{ij}^{e} d\varepsilon_{ij} dz = \int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{ij}^{t} d\varepsilon_{ij} dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\sigma_{ij}^{c} d\varepsilon_{ij} dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{ij}^{b} d\varepsilon_{ij} dz \tag{1.4}$$

Expressing the stresses in the equivalent material, top face, core and bottom face from the constitutive relations Eq. (1.1) into Eq. (1.4) and using some of the relations Eq. (1.3) to cancel out the shear terms we get

$$C_{11}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx}^{2} dz + C_{22}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy}^{2} dz + C_{33}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz}^{2} dz + 2\left(C_{12}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx} d\varepsilon_{yy} dz + C_{23}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy} d\varepsilon_{zz} dz + C_{13}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx} d\varepsilon_{zz} dz\right) =$$

$$C_{11}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{xx}^{2} dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{xx}^{2} dz\right) + C_{22}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{yy}^{2} dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{yy}^{2} dz\right) + C_{33}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{zz}^{2} dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{zz}^{2} dz\right) + 2\left[C_{12}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{xx} d\varepsilon_{yy} dz + \right.\right.$$

$$\left.\left. + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{xx} d\varepsilon_{yy} dz\right) + C_{23}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{yy} d\varepsilon_{zz} dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{yy} d\varepsilon_{zz} dz\right) + C_{13}^{f}\left(\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\varepsilon_{xx} d\varepsilon_{zz} dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\varepsilon_{xx} d\varepsilon_{zz} dz\right)\right] +$$

$$+ C_{11}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx}^{2} dz + C_{22}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy}^{2} dz + C_{33}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz}^{2} dz + 2\left(C_{12}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx} d\varepsilon_{yy} dz + C_{23}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy} d\varepsilon_{zz} dz + C_{13}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx} d\varepsilon_{zz} dz\right) \tag{1.5}$$

Equations (1.3) and (1.5) represent the system of nine linear equations with nine unknowns. After solving this system for the unknowns C_{ij}^{e} we can calculate the stresses in the equivalent material and return them back to the main finite element program.

Note that if the present homogenization procedure is combined with a first order element formulation (strains have at most linear distribution), then due to the fact that the stretching and bending of the shell are uncoupled (symmetric sandwich), all equivalent stresses can be explicitly expressed through very simple relations without having to compute the constitutive constants C_{ij}^{e}. In this case two integration points throughout the thickness will be enough. However, for convenience let us assume three integration points – one at the midplane, and one at each shell surfece – top and bottom. Let us also assume that the strains are distributed through the shell thickness as follows:

$$
\begin{aligned}
d\varepsilon_{xx} &= d\varepsilon_{xx}^{0} + z d\kappa_{xx} & d\varepsilon_{xx} &= d\varepsilon_{xy}^{0} + z d\kappa_{xy} \\
d\varepsilon_{yy} &= d\varepsilon_{yy}^{0} + z d\kappa_{yy} & d\varepsilon_{yz} &= d\varepsilon_{yz}^{0} \\
d\varepsilon_{zz} &= 0 & d\varepsilon_{xz} &= d\varepsilon_{xz}^{0}
\end{aligned}
\tag{1.6}
$$

where $d\varepsilon_{ij}^{0}$ are the strain increments in the midplane and $d\kappa_{ij}$ are the sectional rotation increments. Then $d\varepsilon_{xx}$ and $d\varepsilon_{yy}$ can be decoupled into strains due to axial forces and strains due to bending. In the midsection due to the symmetry of the shell the bending strains will be zero. So, for the midplane integration point we have:

For the core:

$$
\begin{aligned}
d\sigma_{xx}^{c} &= C_{11}^{c} d\varepsilon_{xx} + C_{12}^{c} d\varepsilon_{yy} \\
d\sigma_{yy}^{c} &= C_{12}^{c} d\varepsilon_{xx} + C_{22}^{c} d\varepsilon_{yy} \\
d\sigma_{zz}^{c} &= C_{13}^{c} d\varepsilon_{xx} + C_{23}^{c} d\varepsilon_{yy}
\end{aligned}
\tag{1.7}
$$

For the faces: The stress increments $d\sigma_{ii}^{f}$ (no summation) are calculates similarly to Eq. (1.7). Then for the equivalent homogeneous material the stresses are averaged:

$$
\begin{aligned}
d\sigma_{xx}^{e} &= \frac{2 d\sigma_{xx}^{f} h_f + d\sigma_{xx}^{c} h_c}{2h_f + h_c} \\
d\sigma_{yy}^{e} &= \frac{2 d\sigma_{yy}^{f} h_f + d\sigma_{yy}^{c} h_c}{2h_f + h_c} \\
d\sigma_{zz}^{e} &= \frac{2 d\sigma_{zz}^{f} h_f + d\sigma_{zz}^{c} h_c}{2h_f + h_c}
\end{aligned}
\tag{1.8}
$$

For the in-plane shear stress increments:

$$
\begin{aligned}
d\sigma_{xy}^{c} &= C_{66}^{c} d\varepsilon_{xy} \\
d\sigma_{xy}^{f} &= C_{66}^{f} d\varepsilon_{xy} \\
d\sigma_{xy}^{e} &= \frac{2 d\sigma_{xy}^{f} h_f + d\sigma_{xy}^{c} h_c}{2h_f + h_c}
\end{aligned}
\tag{1.9}
$$

For the integration points outside the midplane, provided we know their *z*-coordinate and knowing the strains in the midsection, $d\varepsilon_{ij}^{0}$: $d\varepsilon_{xx}^{b} = d\varepsilon_{xx} - d\varepsilon_{xx}^{0} = z d\kappa_{xx}$. Here superscript *b* denotes strain due to bending only. Then

$$
\begin{aligned}
d\kappa_{xx} &= \frac{d\varepsilon_{xx} - d\varepsilon_{xx}^{0}}{z} \\
d\sigma_{xx} &= C_{11} d\varepsilon_{xx}^{b} + C_{12} d\varepsilon_{yy}^{b}
\end{aligned}
\tag{1.10}
$$

$$
M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} d\sigma_{xx} z dz = 2\int_{0}^{\frac{h}{2}} d\sigma_{xx} z dz = 2\left[\left(C_{11}^{c} d\kappa_{xx} + C_{12}^{c} d\kappa_{yy}\right)\int_{0}^{\frac{h_c}{2}} z^2 dz + \left(C_{11}^{f} d\kappa_{xx} + C_{12}^{f} d\kappa_{yy}\right)\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_f} z^2 dz \right] =
$$

$$
= \left(C_{11}^{c} d\kappa_{xx} + C_{12}^{c} d\kappa_{yy}\right)\frac{h_c^3}{12} + \left(C_{11}^{f} d\kappa_{xx} + C_{12}^{f} d\kappa_{yy}\right)\left(\frac{3}{2} h_c^2 h_f + 3 h_c h_f^2 + 2 h_f^3\right)
\tag{1.11}
$$

For the equivalent shell the moment is $M_{xy} = \frac{h^2}{3} d\sigma^e_{xx}\left(\frac{h}{2}\right)$. Then $d\sigma^e_{xx}\left(\frac{h}{2}\right) = \frac{3M_{xy}}{h^2}$, where M_{xy} is calculated from Eq. (1.11). The stresses $d\sigma^e_{yy}\left(\frac{h}{2}\right)$ and $d\sigma^e_{xy}\left(\frac{h}{2}\right)$ are calculated in a similar manner.

In order to calculate the transverse shear stresses the corresponding strains in the faces and core are determined based on the strains in the equivalent shell and interface boundary condition – stresses at the face-core interface should be equal:

$$C^c_{55} d\varepsilon^c_{xz} = C^f_{55} d\varepsilon^f_{xz} \tag{1.12}$$

The transverse shear strain in the equivalent material is assumed weighted average of the strains in faces and core:

$$d\varepsilon^e_{xz} = \frac{2d\varepsilon^f_{xz} h_f + d\varepsilon^c_{xz} h_c}{2h_f + h_c} \tag{1.13}$$

The strains $d\varepsilon^f_{xz}$ and $d\varepsilon^c_{xz}$ can be expressed through $d\varepsilon^e_{xz}$ using Eq. (1.12) and Eq. (1.13). Then for the transverse shear stresses in the equivalent shell, core and faces we get

$$d\sigma^e_{xz} = d\sigma^c_{xz} = d\sigma^f_{xz} = \left(\frac{2h_f + h_c}{\frac{2h_f}{C^f_{55}} + \frac{h_c}{C^c_{55}}} \right) d\varepsilon^e_{xz} \tag{1.14}$$

Similarly

$$d\sigma^e_{yz} = d\sigma^c_{yz} = d\sigma^f_{yz} = \left(\frac{2h_f + h_c}{\frac{2h_f}{C^f_{44}} + \frac{h_c}{C^c_{44}}} \right) d\varepsilon^e_{xz} \tag{1.15}$$

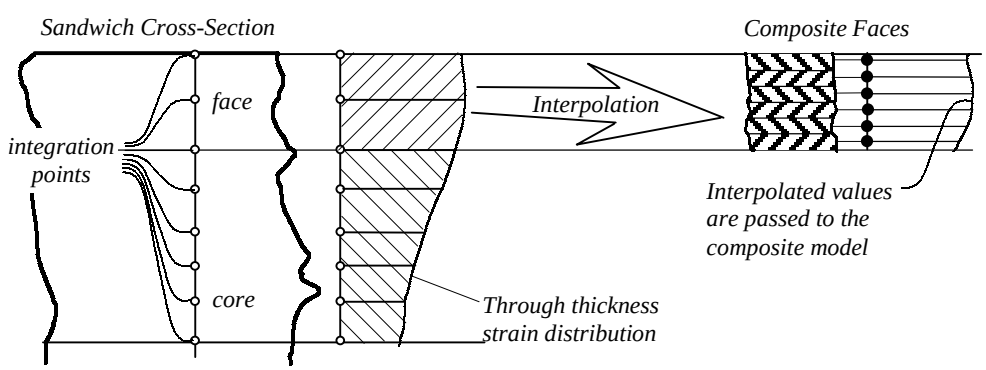


Fig. 1. Sandwich Homogenization Procedure with Composite Faces

As stated above, the sandwich homogenization procedure assumes homogeneous faces. In most practical cases, however, this is not so – faces are usually made of laminated or woven composites. Then a composite homogenization should be performed to be able to get the stresses in an equivalent homogeneous face. A composite micromechanical model for woven fabrics is described here. In order to combine with the composite micromechanical model the relations in the formulation of the sandwich homogenization procedure should be modified as follows:

- Since strains are required by the composite micromechanical model at certain points through the face thickness (e.g. one point for each lamina), the sandwich homogenization procedure interpolates from the

strain values received from the main program to get the strains at the required points through the face thickness (see Fig. 1.). These strains are then passed to the composite micromechanical model and the corresponding stresses are received back;

- The right-hand sides of Eq. (1.3) and Eq. (1.5) are modified to include the integrals involving stresses as in Eq. (1.2) and Eq. (1.4). The values of these integrals are calculated using a numerical integration rule. The system of nine linear equations [Eq. (1.3) and Eq. (1.5)] is then presented in the following form

$$C_{11}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx}dz + C_{12}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy}dz + C_{13}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz}dz =$$

$$\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{xx}^{t}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{xx}^{b}dz + C_{11}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx}dz + C_{12}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy}dz + C_{13}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz}dz$$

$$C_{12}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx}dz + C_{22}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy}dz + C_{23}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz}dz =$$

$$\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{yy}^{t}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{yy}^{b}dz + C_{12}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx}dz + C_{22}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy}dz + C_{23}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz}dz$$

$$C_{13}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx}dz + C_{23}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy}dz + C_{33}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz}dz =$$

$$\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{zz}^{t}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{zz}^{b}dz + C_{13}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx}dz + C_{23}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy}dz + C_{33}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz}dz$$

$$C_{44}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yz}dz = \int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{yz}^{t}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{yz}^{b}dz + C_{44}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yz}dz$$

$$C_{55}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xz}dz = \int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{xz}^{t}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{xz}^{b}dz + C_{55}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xz}dz \qquad (1.3')$$

$$C_{66}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xy}dz = \int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{xy}^{t}dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{xy}^{b}dz + C_{66}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xy}dz$$

$$C_{11}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx} z dz + C_{12}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy} z dz + C_{13}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz} z dz =$$

$$\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{xx}^{t} z dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{xx}^{b} z dz + C_{11}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx} z dz + C_{12}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy} z dz + C_{13}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz} z dz$$

$$C_{12}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx} z dz + C_{22}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy} z dz + C_{23}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz} z dz =$$

$$\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{yy}^{t} z dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{yy}^{b} z dz + C_{12}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx} z dz + C_{22}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy} z dz + C_{23}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz} z dz$$

and

$$C_{11}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx}^{2} dz + C_{22}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy}^{2} dz + C_{33}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{zz}^{2} dz + 2\left(C_{12}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx} d\varepsilon_{yy} dz + C_{23}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{yy} d\varepsilon_{zz} dz + C_{13}^{e}\int_{-\frac{h}{2}}^{\frac{h}{2}} d\varepsilon_{xx} d\varepsilon_{zz} dz \right) =$$

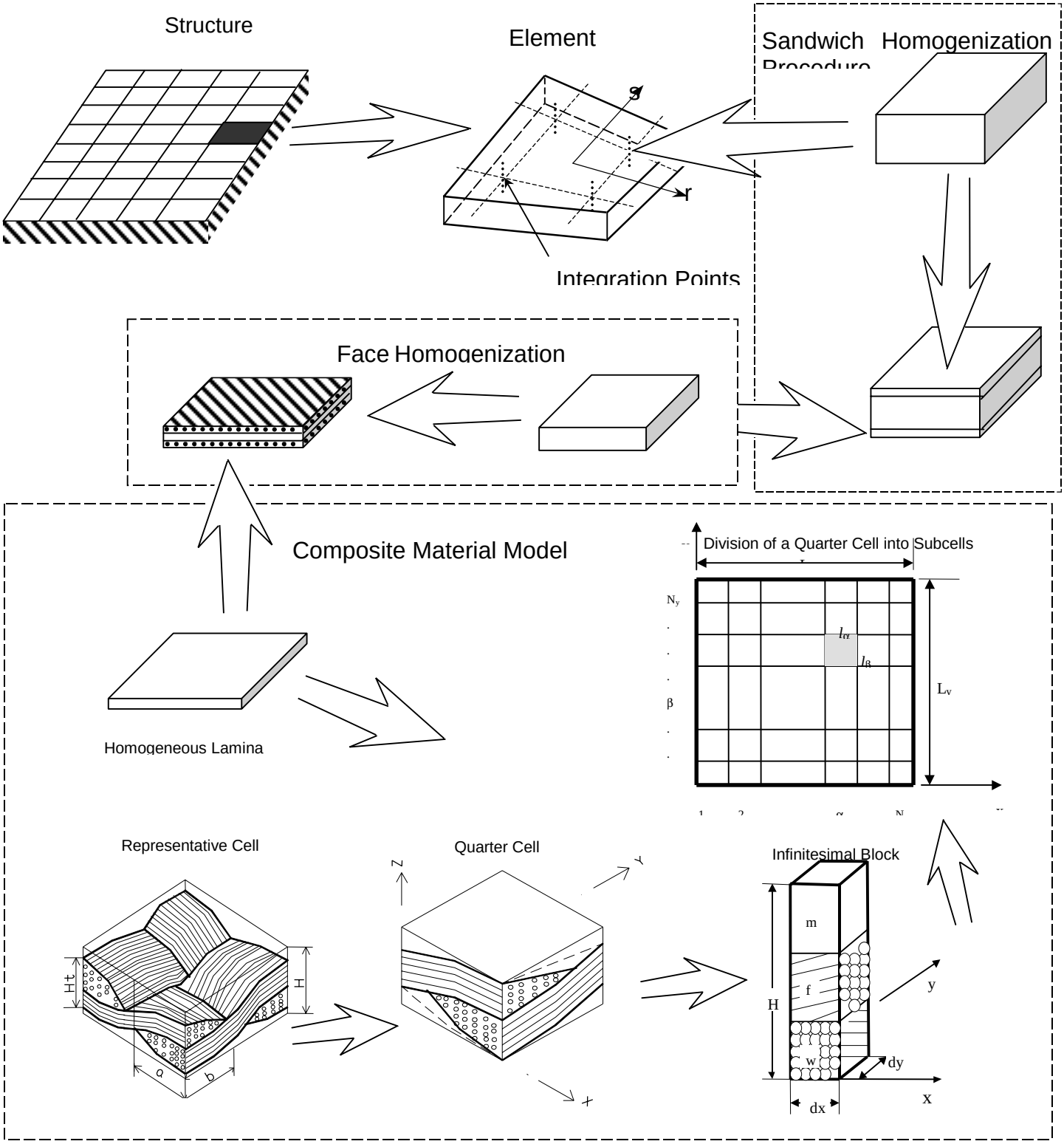
$$\int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{xx}^{t} d\varepsilon_{xx} dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{xx}^{b} d\varepsilon_{xx} dz + \int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{yy}^{t} d\varepsilon_{yy} dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{yy}^{b} d\varepsilon_{yy} dz + \int_{-\frac{h}{2}}^{-\frac{h_c}{2}} d\sigma_{zz}^{t} d\varepsilon_{zz} dz + \int_{\frac{h_c}{2}}^{\frac{h}{2}} d\sigma_{zz}^{b} d\varepsilon_{zz} dz + \tag{1.5'}$$

$$+ C_{11}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx}^{2} dz + C_{22}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy}^{2} dz + C_{33}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{zz}^{2} dz + 2\left(C_{12}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx} d\varepsilon_{yy} dz + C_{23}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{yy} d\varepsilon_{zz} dz + C_{13}^{c}\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} d\varepsilon_{xx} d\varepsilon_{zz} dz \right)$$

This constitutes the sandwich homogenization procedure for composite faces and homogeneous core. When combined with a first order element formulation, relations in Eq. (1.7) through Eq. (1.15) should be changed accordingly. For the midplane integration point instead of using Eq. (1.7) to get the normal stress increments $d\sigma_{ii}^{f}$ (no summation), stresses are calculated by the composite micromechanics homogenization model. Then for the equivalent homogeneous material the stresses are averaged according to Eq. (1.8). Similarly the in-plane shear stresses are calculated. For the integration points outside the midplane, and composite faces, to calculate M_{xy}, we need the stresses in the upper and lower surface of the faces. Then

$$M_{xy} = \left(C_{11}^{c} d\kappa_{xx} + C_{12}^{c} d\kappa_{yy}\right)\frac{h_c^3}{12} + h_f \frac{d\sigma_{xx}^{f}\left(\frac{h}{2}\right) + d\sigma_{xx}^{f}\left(\frac{h_c}{2}\right)}{2}\left\{\frac{h_c}{2} + \frac{h_f}{3}\cdot\frac{2d\sigma_{xx}^{f}\left(\frac{h}{2}\right) + d\sigma_{xx}^{f}\left(\frac{h_c}{2}\right)}{d\sigma_{xx}^{f}\left(\frac{h}{2}\right) + d\sigma_{xx}^{f}\left(\frac{h_c}{2}\right)}\right\} \tag{1.11'}$$

Fig.2. The Sandwich Homogenization Procedure and the Composite Micromechanics Model



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To calculate the stresses in the upper and lower face surface first the strains are calculated for each composite layer and are passed to the composite micromechanics model which then returns the corresponding stresses. Using these stress values extrapolation is performed to determine $d\sigma_{xx}^{f}\left(\frac{h}{2}\right)$ and $d\sigma_{xx}^{f}\left(\frac{h_c}{2}\right)$. The value for M_{xy} is then used to calculate $d\sigma_{xx}^{e}\left(\frac{h}{2}\right)=\frac{3M_{xy}}{h^2}$. The stresses $d\sigma_{yy}^{e}\left(\frac{h}{2}\right)$ and $d\sigma_{xy}^{e}\left(\frac{h}{2}\right)$ are calculated in a similar manner. Fig. 2. presents a scheme illustrating how the sandwich homogenization procedure and the composite micromechanics model are organized and their interrelationship and interface to the main analysis program.

WOVEN COMPOSITE FACES

As already stated, to be able to analyze sandwich shells with composite faces, the present sandwich homogenization procedure should be combined with a composite homogenization model. A micromechanics-based model for woven composites developed by Tabiei and Jiang (1998) is presented here. Within the model a representative volume cell is assumed. Using the iso-stress and iso-strain assumptions the constitutive equations are averaged along the thickness direction. The cell is then divided into many subcells and an averaging is performed again by assuming uniform stress distribution in each subcell to obtain the effective stress-strain relations of the subcell. The stresses and strains within the subcells are combined to yield the effective stresses and strains in the representative cell. Then this information is passed to the sandwich homogenization procedure. Also, global stresses and strains are communicated to the representative cell and subsequently distributed to each subcell. Once stresses and strains are associated to a subcell they can be distributed to each constituent of the subcell i.e. fill, warp, and resin.

Mathematical Formulation

Any non-hybrid plain weave fabric composite laminae can be represented by double-periodic representative volume cells, as shown in Fig. 2. The representative volume cell typically consists of two sets of interlaced yarns, known as fill and warp threads, and resins. The fill and warp threads, might be pure fibers or mixture of fibers and resins, are assumed to be homogenous and transversely isotropic. The resin is homogenous and isotropic. Given a representative volume cell, the global average stresses and strains can be written as:

$$\left\{\bar{\sigma}\right\}=\frac{1}{V}\iiint\{\sigma\}dv \quad \text{and} \quad \left\{\bar{\varepsilon}\right\}=\frac{1}{V}\iiint\{\varepsilon\}dv \tag{2.1}$$

The question that must be answered is that, given the average global stresses (or strains), what are the global strains (or stresses). For this purpose, a micromechaics-based model is developed on the basis of assumptions related to local stresses and strains.

Effective Stress-Strain Relations For An Infinitesimal Element

Consider a typical infinitesimal block of the representative cell with dx in length, dy in width and H in height, as shown in Fig. 2. This infinitesimal element generally consists of three different materials with $\bar{t}_m$, $\bar{t}_f$ and $\bar{t}_w$ volume fractions of matrix, fill and warp respectively. The material coordinate system of fills and warps may not coincide with the global coordinate system. Assuming that the in-plane relations for these three materials are in a parallel arrangement, the following equations are derived for incremental strains and stresses:

$$d\left\{\varepsilon_i\right\}=d\left\{\varepsilon_i\right\}_k \qquad \text{(parallel)} \tag{2.2-a}$$

$$d\left\{\sigma_i\right\}=\sum_{k=m,f,w}\bar{t}_k\, d\left\{\sigma_i\right\}_k \qquad \text{(parallel)} \tag{2.2-b}$$

where the subscript k denotes constituents with m for matrix, f for fill and w for warp. $\{\varepsilon_i\}$ and $\{\sigma_i\}$ are the in-plane strains and stresses respectively and the subscript $i = xx, yy, xy$. In the material coordinate system, the constitutive laws for matrix, fill and warp are all written as

$$\begin{Bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\gamma_{12} \\ d\varepsilon_{33} \\ d\gamma_{13} \\ d\gamma_{23} \end{Bmatrix}_k = \begin{bmatrix} S_{11}^t & S_{12}^t & 0 & S_{13}^t & 0 & 0 \\ S_{12}^t & S_{22}^t & 0 & S_{23}^t & 0 & 0 \\ 0 & 0 & S_{66}^t & 0 & 0 & 0 \\ S_{13}^t & S_{23}^t & 0 & S_{33}^t & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55}^t & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{44}^t \end{bmatrix}_k \begin{Bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{12} \\ d\sigma_{33} \\ d\sigma_{13} \\ d\sigma_{23} \end{Bmatrix}_k \tag{2.3}$$

where the compliance components with superscript t represent the tangential ones. For symbolic derivation thereafter, Eq. (2.3) can be simply written as

$$\begin{Bmatrix} d\varepsilon_i^* \\ - \\ d\varepsilon_o^* \end{Bmatrix}_k = \begin{bmatrix} S_i & | & S_{io} \\ - & - & - \\ S_{oi} & | & S_o \end{bmatrix}_k^t \begin{Bmatrix} d\sigma_i^* \\ - \\ d\sigma_o^* \end{Bmatrix}_k \tag{2.4}$$

where the subscript i= 11, 22, 12 and o = 33, 13, 23 respectively. In the global coordinate system, the constitutive relations can be written as:

$$\begin{Bmatrix} d\varepsilon_i \\ - \\ d\varepsilon_o \end{Bmatrix}_k = \begin{bmatrix} \psi_i & | & \psi_{io} \\ - & - & - \\ \psi_{oi} & | & \psi_o \end{bmatrix}_k^t \begin{Bmatrix} d\sigma_i \\ - \\ d\sigma_o \end{Bmatrix}_k \tag{2.5}$$

with

$$\begin{bmatrix} \psi_i & | & \psi_{io} \\ - & - & - \\ \psi_{oi} & | & \psi_o \end{bmatrix}_k^t = [T_1]_k \begin{bmatrix} S_i & | & S_{io} \\ - & - & - \\ S_{oi} & | & S_o \end{bmatrix}_k^t [T_1]_k^T \tag{2.6}$$

The expression of the transformation matrix $[T_1]_k$ can be found in Appendix I of Whitcomb (1991).

In application of finite element method for analysis of the global behavior of thin-walled structures made of woven fabric composites, shell elements are usually employed. The required constitutive laws are generally written as follows:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \tag{2.7}$$

and

$$\begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix} = \begin{bmatrix} S_{44} & 0 \\ 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \sigma_{23} \\ \sigma_{13} \end{Bmatrix} \tag{2.8}$$

In these equations the global normal stress in thickness direction is ignored. During the incremental-iterative solution scheme, as used in finite element analysis of nonlinear problems, a change in the nodal displacements takes place. The displacement increment causes an increment of strain $\Delta\{\bar{\varepsilon}_i\}$ at a material point. The material model is required to calculate the tangential stiffness matrix and the incremental stress $\Delta\{\bar{\sigma}_i\}$. In this investigation, stiffnesses, strains, and stresses are tracked at the material points within each element. This information is provided by the woven composite material model. The woven heterogeneous nature of the material is hidden from the sandwich homogenization subroutine. Figure 2 shows a schematic of the micromechanical model and the interface with the sandwich homogenization procedure. Due to the symmetry of the representative unit cell, only quarter cell is considered. This quarter cell represents the same mechanical properties as the whole cell. The quarter cell is further divided into subcells. Each subcell is represented by an infinitesimal block as shown in Fig. 2. The actual woven composite is replaced by an equivalent homogenous material whose properties are determined by requiring that the actual material and the equivalent material behave in the same way when subjected to certain stresses or strains. The interface consists of stresses and strains transfer between the material model and the analysis code. The sandwich homogenization procedure only sees this equivalent homogenous anisotropic material.

Neglecting normal stress in thickness direction and the coupling among in-plane normal components and

transverse shear components, the in-plane stress-strain relations for off-axis fill tows, warp tows and matrix can be written as:

(1) Off-axis fill

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}_f = \begin{bmatrix} \psi_{11} & \psi_{12} & 0 \\ \psi_{12} & \psi_{22} & 0 \\ 0 & 0 & \psi_{66} \end{bmatrix}_f \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}_f \tag{2.9}$$

where

$$\begin{aligned} (\psi_{11})_f &= c^4 (S_{11})_f + 2c^2 s^2 (S_{13})_f + s^4 (S_{33})_f + c^2 s^2 (S_{55})_f \\ (\psi_{12})_f &= c^2 (S_{12})_f + s^2 (S_{23})_f \\ (\psi_{22})_f &= (S_{22})_f \\ (\psi_{66})_f &= c^2 (S_{66})_f + s^2 (S_{44})_f \qquad \text{with } c = \cos(\theta_f) \text{ and } s = \sin(\theta_f) \end{aligned}$$

(2) Off-axis warp

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}_w = \begin{bmatrix} \psi_{11} & \psi_{12} & 0 \\ \psi_{12} & \psi_{22} & 0 \\ 0 & 0 & \psi_{66} \end{bmatrix}_w \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}_w \tag{2.10}$$

where

$$\begin{aligned} (\psi_{22})_w &= c^4 (S_{11})_w + 2c^2 s^2 (S_{13})_w + s^4 (S_{33})_w + c^2 s^2 (S_{55})_w \\ (\psi_{12})_w &= c^2 (S_{12})_w + s^2 (S_{23})_w \\ (\psi_{11})_w &= (S_{22})_w \\ (\psi_{66})_w &= c^2 (S_{66})_w + s^2 (S_{44})_w \qquad \text{with } c = \cos(\theta_w) \text{ and} \end{aligned}$$

(3) Matrix

(2.11)

Eq. (2.5) can be written for the inplane stresses in the following matrix form as:

(2.12)

Incorporating Eqs. (2.2) and (2.12), the following Eqs. can be derived

(2.13)

where

(2.14)

Finally, the effective stress-strain relations for the infinitesimal element are obtained by rearranging Eq. (2.13) as follows:

(2.15)

where

(2.16)

Average Stress-Strain Relations For A Subcell

The whole representative volume cell is divided into many subcells. It is assumed that stresses are uniformly distributed within each subcell. The incremental average stress-strain relations for a subcell can be obtained by performing integration of Eq. (2.15), i.e.:

(2.17)

with

(2.18)

From Eqs. (2.9)-(2.11) the incremental form of in-plane stress-strain relations of constituents may be written as:

(2.19)

Once the incremental forms of the stress-strain relations of the constituents are obtained from Eq. (2.19), the in-plane relations for a subcell can be derived from Eq. (2.17) and are denoted by:

(2.20)

In this procedure it is assumed that the average in-plane strains and stresses among subcells have the following relationships:

(,) (2.21-a)

(, ,) (2.21-b)

(,) (2.21-c)

(,) (2.21-d)

( ) (2.21-e)

( ) (2.21-f)

(2.21-g)

where the quantities with "bar" denote the incremental average strain stress components of the whole cell. The incremental average stresses induced by increment average strains are expressed as:

(2.22)

Equation (2.21) together with Eq. (2.20) provides sufficient information to distribute the incremental average strains to each subcell. Once the average strains in each subcell is known, the incremental average stresses of the cell can be obtained by Eqs. (2.20) and (2.22). Simultaneously, one can obtain the tangential stiffness matrix. In the next section the calculation procedure will be provided. Now Eqs. (2.20) and (2.21) form a simultaneous linear system of equations, with incremental strains of each subcell as unknown and incremental average strains of the cell as known, presented by the following equation:

(2.23)

with

(2.24)

From Eq. (2.23), the incremental strains of each subcell can be solved for by the following:

(2.25)

The partitioned form of Eq. (2.25) is

(and) (2.26)

By combining Eqs. (2.20), (2.21) and (2.26), the incremental average stresses can be obtained:

(2.27)

or

(2.28)

with

(2.29)

Equation (2.29) provides the total tangential stiffness matrix for the average in-plane stress-strain relations. Once we know the average stresses in the facing, the sandwich homogenization procedure can be completed.

CONCLUSIONS

A sandwich homogenization procedure is presented and a solution scheme is outlined for implementation in structural finite element codes. The proposed sandwich homogenization procedure makes possible the analysis of sandwich shell structures by means of standard finite elements for homogeneous materials. Furthermore, when combined with a composite homogenization model it can be used to investigate the behavior of sandwich shells with composite faces. In the present work the sandwich homogenization procedure is coupled with a new efficient micromechanics-based model for woven composites. The proposed homogenization procedure can be combined with all finite element shell formulations (based on first order Reissner-Mindlin theory or on higher order shell theory) depending on the available computational resources and on the required accuracy and efficiency. The presented methodology is most suitable for including material nonlinear behavior of sandwich constituents. The advantage of this procedure is the ability to use standard shell finite element formulation and save significant effort of developing new sandwich finite element shells.

ACKNOWLEDGEMENTS

The research in this study is supported by a Grant from the Airforce Office of Scientific Research (grant number F49620-98-1-0384). Their support is gratefully acknowledged.

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Symposium of Recent Advances in Mechanics of Aerospace Structures and Materials, AD-Vol 56, pp. 219-235, ASME winter meeting, Anaheim, CA, Nov. 1998.

A LOCAL/GLOBAL SANDWICH HOMOGENIZATION PROCEDURE FOR FINITE ELEMENT SIMULATION

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ABSTRACT

A new finite element approach to sandwich shells is proposed. It uses existing shell finite elements formulated for homogeneous shells. The "sandwich" nature of the problem is hidden from the main finite element program. Based on several assumptions the proposed homogenization procedure calculates stress increments in a homogeneous fictitious material, called "equivalent", which correspond to the strain increments in the equivalent material points. The stresses in the equivalent material are calculated based on the stress and strain fields in the sandwich layers, which are determined from the incoming strain field for the equivalent material. This procedure is most suitable for including strain softening and other material nonlinear behavior. The approach can be combined with shell elements formulated for homogeneous materials, based on Reissner-Mindlin shell theory or with elements based on a higher order shell theory. It avoids the necessity of formulating special shell elements for sandwich constructions, which in most cases, due to their large number of degrees of freedom, significantly decrease the computational efficiency of the finite element analysis. The sandwich homogenization procedure is combined with a composite micromechanics-based model for woven composites to analyze sandwich shells with woven fabric faces.

Symposium of Recent Advances in Mechanics of Aerospace Structures and Materials, AD-Vol 56, pp. 219-235, ASME winter meeting, Anaheim, CA, Nov. 1998.