Solution to Problem 6.8

(a) Forming the closed loop transfer function’s denominator, we get:

\[ s^3 + 3 s^2 + (5 - K) s + (2 K + 3) \]

We see that all the coefficients will be the same sign only if K < 5 and if K > -3/2. To see if there are further conditions on K for closed loop stability, we construct the Routh array:

\[
\begin{array}{ccc}
 s^3 & 1 & 5 - K \\
 s^2 & 3 & 2 K + 3 \\
 s^1 & a & \\
 s^0 & 2 K + 3 \\
\end{array}
\]

where \( a = \frac{3 (5 - K) - 1 (2 K + 3)}{3} = \frac{12 - 5 K}{3} \). To avoid sign changes in the first column (and hence to have no right half-plane poles), we need \( a > 0 \), which implies \( K < 12/5 \) (which is a tighter constraint than \( K < 5 \) from above). So, the range of K for closed loop stability is \( -3/2 < K < 12/5 \).

(b) When \( K < -3/2 \), the last element in the first column of the Routh array becomes negative. There is then one sign change in the first column, and therefore there is exactly one right half-plane closed loop pole. Since there is only one, it cannot be a pair of complex conjugates, and hence it must be a positive real pole. The time response corresponding to this includes an exponential divergence term. In ship-steering terms, the ship will diverge from its desired course to either the left or the right. (Should show a sketch.)

(c) When \( K > 12/5 \), the element \( a \) in the first column is negative, and so there are two sign changes in the first column of the Routh array, meaning there are exactly two unstable closed loop poles. If \( K \) is just slightly larger than 12/5, these unstable poles will be a complex conjugate pair just to the right of the imaginary axis. In this case, the ship’s course will oscillate from left to right with increasing magnitude.

EXTRA CREDIT: When K is half its upper limit for stability, we have \( K = 6/5 = 1.2 \). This yields closed loop poles at -2.36 (OK, stable and corresponding response term converges in about 2 time units) and at -0.320 +/- j1.48. This latter pair corresponds to a slowly converging oscillatory term with a settling time of more than 12 time units, over which time it oscillates about six cycles. Furthermore, the closed loop zero at +2 makes the oscillations even worse, besides making the response start out the “wrong” way. This is not a boat I would want to ride in! In trying to find the “best” K by trial-and-error, the problem is that no positive K yields a closed loop damping ratio any higher than 0.58, which is still fairly low (though it does produce a reasonably good response). Negative K (above the threshold for stability) produces a larger damping ratio, but also produces a third real closed loop pole that gets close to the origin, and
hence produces a very slow exponential term in the response. Furthermore, negative K causes the steady state output to be opposite in sign to the input, so the ship’s helmsperson will have to learn to give backwards commands (or some kind of gearing will have to be rigged to reverse the command directions).

Solution to Problem 6.11

(a) Answer is given.

(b) Closed loop denominator: \( s^3 + 6 s^2 + (5 + 4.5K_D)s + (4.5K_p) \). Clearly \( K_P > 0 \) is required for stability, as is \( K_D > -10/9 \). The Routh array:

\[
\begin{array}{cccc}
  s^3 & 1 & 5 + 4.5K_D \\
  s^2 & 6 & 4.5K_p \\
  s^1 & a & \\
  s^0 & 4.5K_p \\
\end{array}
\]

where \( a = \frac{6(5 + 4.5K_D) - 4.5K_P}{6} \). To avoid sign changes in the first column, we must have \( K_P < \frac{(30 + 27K_D - 4.5K_P)}{4.5} \). Therefore, the range on \( K_p \) for stability is \( 0 < K_P < \frac{(30 + 27K_D)}{4.5} \). This range will not exist unless \( K_D > -10/9 \).

(c) When \( K_D = 1 \), \( a \) becomes \( \frac{57 - 4.5K_P}{6} \). If this is zero, we will have all the elements in the \( s^1 \) row of the Routh array equal to zero. This will yield poles at the roots of the auxiliary polynomial formed from the \( s^2 \) row, i.e. at the roots of \( 6s^2 + 4.5K_p \), which will be imaginary and hence will yield an undamped oscillatory response term. The term \( a \) is zero when \( K_P = \frac{57}{4.5} = 12.67 \). This yields closed loop poles at \( \pm j 3.082 \), which yields an undamped oscillation in the response at 3.082 rad/sec (really rad/(time unit)).