Chapter 10: Compressible Flow with Heat Addition or Loss

10.1 Introduction:

- In previous chapters adiabatic flows were investigated with and without area change or friction.
- In this chapter heat addition or heat loss in a 1D flow of a compressible gas will be examined.
10.2 Constant Area Frictionless Flow w/Heat Transfer: Rayleigh Line Flow

- Assume a steady frictionless flow, through a constant area duct with heat addition, \( q \) (J/kg).

\[ \frac{dq}{\rho} = \frac{\rho}{\rho + d\rho} \frac{dV}{V} \]

- The steady state conservation of mass (recall \( A = \text{constant} \))

\[ \rho V = (\rho + d\rho)(V + dV) \]

- Integrate and obtain

\[ \ln \rho + \ln V = \text{const} \]

- Raise the above to the exp power:

\[ e^{\ln \rho V} = e^{\text{const}} \]

\[ \rho V = \text{const} \]

- The steady state momentum eq is simplified to

\[ dp + \rho V dV = 0 \]

(recall the only forces acting on control volume are pressure forces) now rearrange,
10.2 Constant Area Frictionless Flow w/Heat Transfer: Rayleigh Line Flow

- So
  \[ dp + \rho \frac{dV^2}{2} = 0 \]

- Integrating,
  \[ p + \int \rho \frac{dV^2}{2} = 0 \]

- This can not be simply integrated because \( r \) is not a constant (ie., compressible flow).
  \[ p + \int \rho \frac{2V}{2} \frac{dV}{2} = 0 \]

- Instead we substitute \( \nu = \) constant from the continuity eq and integrate,
  \[ \int dp + \text{const} \int dV = \text{const} \]

Replace the first constant by \( \nu \) and
\[ p + \rho \nu^2 = \text{const} \quad (10.2) \]

- The steady state energy eq simplified is
  \[ dq = VdV + dh \quad (10.3) \]

- For a perfect gas
  \[ dh = c_p dT \]

- So;
  \[ dq = VdV + c_p dT \]
10.2 Constant Area Frictionless Flow w/Heat Transfer: Rayleigh Line Flow

- So

\[ dq = d\left(\frac{V^2}{2}\right) + c_p dT \]  

(10.4)

- Using the definition of stagnation temperature;

\[ T_t = T + \frac{V^2}{2c_p} \]

or

\[ (T_t - T)c_p = \frac{V^2}{2} \]

- Therefore,

\[ dq = d(T_t - T)c_p + c_p dT \]

\[ dq = c_p dT_t \]

(10.5)

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10.2 Constant Area Frictionless Flow w/Heat Transfer: Rayleigh Line Flow

- Taking eq 10.2 and using the perfect gas law: \[ p = \rho RT \]

- Hence,

\[ p + \rho V^2 = \text{const} \]

(10.2)

\[ p + \frac{p}{RT} V^2 = \text{const} \]

\[ p \left(1 + \frac{V^2}{RT}\right) = \text{const} \]

- Since \( a^2 = \gamma RT \)

\[ p \left(1 + \frac{\gamma V^2}{a^2}\right) = \text{const} \]
10.2 Constant Area Frictionless Flow w/Heat Transfer: Rayleigh Line Flow

\[ p(1 + \gamma M^2) = \text{const} \quad (10.6) \]

- Also from the perfect gas law,
  \[ T = \frac{p}{\rho R} \]

  and,
  \[ \rho = \frac{\dot{m}}{AV} \]

  where \[ V = M \sqrt{\gamma RT} \]

- Therefore
  \[ T = \frac{pAV}{\dot{m}R} = \frac{pAM\sqrt{\gamma RT}}{\dot{m}R} \]

  \[ \sqrt{T} = pM \ast \text{const} \]

10.2 Constant Area Frictionless Flow w/Heat Transfer: Rayleigh Line Flow

or

\[ T = p^2 M^2 \ast \text{const}^2 \]

- Rearranging eq 10.6,
  \[ p = \frac{\text{const}}{1 + \gamma M^2} \]

- Therefore
  \[ T = \frac{M^2}{\left(1 + \gamma M^2\right)^2} \ast \text{Const} \quad (10.7) \]
10.2 Constant Area Frictionless Flow w/Heat Transfer: Rayleigh Line Flow

- Apply eqs 10.6 and 10.7 to the control volume with heat addition, \( q \)

- Therefore

\[
p_1 \left(1 + \gamma M_1^2\right) = p_2 \left(1 + \gamma M_2^2\right) \tag{10.8}
\]

and

\[
\frac{T_1 \left(1 + \gamma M_1^2\right)^2}{M_1^2} = \frac{T_2 \left(1 + \gamma M_2^2\right)^2}{M_2^2} \tag{10.9}
\]

- Also integrating (eq 10.5) for a perfect gas with constant specific heats one obtains;

\[
q = c_p \left(T_{t_2} - T_{t_1}\right) \tag{10.5}
\]

- To facilitate the tabulation of these expressions, let state 2 be a reference state @ which \( M = 1 \) (use the * state)

\[
\frac{p}{p^*} = \frac{1 + \gamma}{1 + \gamma M^2} \tag{10.11}
\]

\[
\frac{T}{T^*} = \left(\frac{1 + \gamma}{1 + \gamma M^2}\right)^{\gamma/\gamma - 1} \tag{10.12}
\]

- Now solve for \( V/V^* \)

\[
\frac{V}{V^*} = \frac{\rho^*}{\rho} = \frac{p^* T}{p T^*}
\]
10.2 Constant Area Frictionless Flow w/Heat Transfer: Rayleigh Line Flow

- Substitute eqs 10.11 and 12
  \[ \frac{V}{V^*} = \frac{1 + \gamma M^2}{1 + \gamma} \left( 1 + \frac{\gamma}{2} M^2 \right) \]

or
  \[ \frac{V}{V^*} = \frac{(1 + \gamma) M^2}{1 + \gamma M^2} \]  
  \[ \text{(10.13)} \]

- Also
  \[ T_i = T \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \]

  \[ \frac{T_i}{T_i^*} = \left( \frac{1 + \gamma}{1 + \gamma M^2} \right)^2 \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \]
  \[ \text{(10.14)} \]

and we know
  \[ p_i = p \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma}{\gamma - 1}} \]

- Therefore,
  \[ \frac{p_i}{p_i^*} = \left( \frac{1 + \gamma}{1 + \gamma M^2} \right)^{\frac{\gamma}{\gamma - 1}} \]
  \[ \text{(10.15)} \]

- Eqs 10.11 to 10.15 are tabulated in the Rayleigh Line flow tables (for \( \gamma = 1.4 \)).
10.2 Constant Area Frictionless Flow w/Heat Transfer: Rayleigh Line Flow

Example 10.1: Air is flowing in a constant area duct at M = 0.20, with a static temperature & pressure of 300 K and 100 kPa. Heat is then added to the flow at a rate of 50 kJ/kg. Assume frictionless flow with air behaving as a perfect gas with constant specific heat, 

\[ c_p = 1.004 \text{kJ/kgK} \]

Determine the final state of the air.

\[ q = 50 \text{ kJ/kg} \]

\[ M_1 = 0.20 \quad T_1 = 300 \text{ K} \quad p_1 = 100 \text{ kPa} \]

\[ M_2 = ? \quad T_2 = ? \quad p_2 = ? \]
10.2 Constant Area Frictionless Flow w/Heat Transfer: Rayleigh Line Flow
10.3 The Rayleigh Line

- The locus of states indicating the effect of heat addition on $M_\#, \text{ for a given mass flow, is plotted on a } T\delta \text{ diagram and is called a Rayleigh line.}$

- For a perfect gas,

$$ds = c_p \frac{dT}{T} - R \frac{dp}{p}$$

(1.30)

- Integrating between some arbitrary point and the reference state (the * state where $M = 1$), assuming $c_p = \text{ const.}$

$$s - s^* = c_p \ln \frac{T}{T^*} - R \ln \frac{p}{p^*}$$

(10.16)

- From eqs 10.11 and 10.12 express $p/p^*$ in terms of $T/T^*$,

$$\frac{p}{p^*} = \frac{1 + \gamma}{1 + \gamma M^2} \quad \text{and} \quad \frac{T}{T^*} = \frac{(1 + \gamma)^2 M^2}{(1 + \gamma M^2)^2}$$
10.3 The Rayleigh Line

• Expressing the M# in terms of the pressure ratio

\[ M^2 = \frac{p^*}{p} \left( \frac{1 + \gamma}{\gamma} \right) - \frac{1}{\gamma} \]

• Combining eqs 10.11 & 10.12 one directly obtains either,

\[ \frac{T}{T^*} = \left( \frac{p^*}{p} \right)^2 M^2 \quad \text{or} \quad \frac{p}{p^*} = \frac{1}{M} \sqrt{\frac{T}{T^*}} \]

or substituting the pressure ratio for M, from above

\[ \frac{p}{p^*} \sqrt{\frac{1 + \gamma}{\gamma} \left( \frac{p^*}{p} \right) - 1} = \sqrt{\frac{T}{T^*}} \]

• Squaring;

\[ \frac{\gamma T}{T^*} = \left( \frac{p}{p^*} \right)^2 \left( (1 + \gamma) \left( \frac{p^*}{p} \right) - 1 \right) \]

10.3 The Rayleigh Line

• Simplifying and arranging in quadratic form gives

\[ \left( \frac{p}{p^*} \right)^2 - (1 + \gamma) \left( \frac{p}{p^*} \right) + \frac{\gamma T}{T^*} = 0 \]

or

\[ \frac{p}{p^*} = 1 + \gamma \pm \sqrt{\frac{(1 + \gamma)^2 - (4\gamma T/T^*)}{2}} \]

• So eq10.16 becomes the relation for Rayleigh Line flow,

\[ \frac{s - s^*}{c_p} = \ln \frac{T}{T^*} - \frac{1}{\gamma} \ln \frac{\gamma + 1 \pm \sqrt{(\gamma + 1)^2 - (4\gamma T/T^*)}}{2} \]

(10.17)

• See Figure 10.4
10.3 The Rayleigh Line

Note:
1) Point A, \( \frac{ds}{dT} = 0 \) and \( M = 1 \) (found using the following analysis)

Continuity;
\[
\frac{dp}{\rho} + \frac{dV}{V} = 0
\] (10.1)

Cons of Momentum;
\[
dp + \rho V dV = 0
\] (10.2)

10.3 The Rayleigh Line

- Combining,
\[
dp - \frac{dp}{dV} V^2 dV = 0
\]
\[
dp - V^2 d\rho = 0
\]
\[
dp = V^2 d\rho
\]
\[
V^2 = \frac{dp}{d\rho}
\] @ \( ds = 0 \)

- Recall from eq 2.4, sound wave processes are isentropic.

Also recall
\[
\frac{\partial p}{\partial p}_{ds=0} = a^2
\]

- Therefore, \( a^2 = V^2 \) and hence \( M = 1 \)
2) Point B, point of tangency where $T/T^*$ is a maximum (however it has no physical reason).

- To determine this maximum, set

$$\frac{dT}{dT^*} = 0$$

- Differentiating Eq. (10.12):

$$\frac{T}{T^*} = \left(\frac{1 + \gamma}{\gamma M^2}\right)^2$$

one obtains

$$\frac{(1 + \gamma M^2)^2}{(1 + \gamma M^2)^2} \cdot 2 M - (1 + \gamma M^2)^2 \cdot 2\gamma M M^2 = 0$$

Simplifying and setting the numerator = 0 yields,

$$1 + \gamma M^2 = 2\gamma M^2$$

$$M^2 = \frac{1}{\gamma}$$

or

$$M = \sqrt{\frac{1}{\gamma}}$$

for air is 0.85

(10.18)

- Substitute this value of back into eq.10.12 & obtain maximum value for air,

$$\frac{T}{T^*}_{MAX} = \left(\frac{1 + \gamma}{\gamma}\right)^2 \frac{1}{4 \gamma} = 1.029$$

(10.19)

- Now suppose heat is added reversibly to a flow in a constant area duct:
  i) Starting at point “a” we have subsonic flow,
  ii) We know $ds = \frac{dq}{T}$, so the addition of heat will result in an increase in entropy of the fluid. So one would move to the right on the following Ts diagram.
10.3 The Rayleigh Line

iii) Point B: the location where \( T/T^* \) is maximum, the flow is still subsonic & \( T_t \) is increasing as the entropy increases to Pt “A”.

iv) Point “A”, where \( M = 1 \), represents a state of maximum entropy for a given \( \dot{m} \) and any further heat addition would reduce the mass flow rate in the duct, that is one would jump to another Rayleigh line of lower \( \dot{m} \).

So, there is a maximum amount of heat that can be added to a duct flow, this maximum is limited by attainment of \( M = 1 \).

Therefore, duct flow can be choked by either heat or friction.

v) As \( M \) is increased to 1, the pressure drops and the temperature initially increases to \( \frac{T}{T_{\text{MAX}}} \) at “B” and then decreases to pt “A.”

For cooling the reverse occurs.

vi) For supersonic flow, as heat is added the \( M \) decreases to 1. Thus, from the momentum eq the static pressure increases to \( p^* \), and the static temperature increase to \( T^* \)

Changes which are brought about by heat addition (Rayleigh line flow) reveal itself by an increased \( T_r \).

This is analogous to methods used in Fanno flow, where \( \frac{fL_{\text{MAX}}}{D} \) was the critical parameter.
10.3 The Rayleigh Line

- For Rayleigh flow it can be seen that the same overall changes in properties occur whether the heat is added:
  i. at a single cross section, as in a flame front or
  ii. distributed over the length of the control volume as a heat exchanger.

\[ \dot{q} \]

\[ \dot{m} \]

\( \gamma = 1.4 \)  
\( c_p = 1.004 \text{ kJ/kgK} \)

Also neglect friction.

Example 10.2:
Air enters a ramjet combustor with a velocity of 100 m/s and static \( T = 400 \text{ K} \).

Determine the maximum amount of heat that can be added in the combustion chamber without reducing the mass flow rate.

For this \( \dot{q}_{\text{MAX}} \) (in Kw), find the fuel/air (F/A) ratio. If the F/A ratio were to be increased by 10 %, determine the reduction in \( n_H \) for the same inlet stagnation pressure and temperature.

Assume the heating value (H.V.) of the fuel to be 40 MJ/kg, neglect the fuel flow rate in comparison to the air flow rate, and assume the air behaves as a perfect gas with constant \( c_p \).
Example 10.3:

N₂ enters an uninsulated duct at \( M = 2 \), with a \( T_t \) and \( p_t \) of 1000 K and 1.4 MPa. Heat is lost from the N₂ to the outside air @ 20°C, with mean overall heat transfer coefficient \( (h) \) between fluid and air equal to 60 W/m²K. A differential heat loss from the N₂ can be expressed as

\[
dq = h \, dA \, (T_t - T_a)
\]

with \( T_a \), the ambient air temperature and \( dA \), a differential area normal to the direction of heat flow. The duct \( D = 5 \) cm and the length is 2m.

Determine the outlet \( T_t \),
M#, and % change in \( p_t \).

Neglect friction and let \( c_p = 1.038 \) kJ/kg K
10.3 The Rayleigh Line

Example 10.4: A constant area duct is connected to a high pressure air reservoir through a converging nozzle. The duct walls are heated and supply 250 kJ/kg of air passing through the duct. If the reservoir p & T is 750 kPa & 300 K respectively, and the system back pressure 300 kPa.

Determine if the duct is choked. Also, find the mass flow rate of air passing through the duct.

The duct is 1.2 m in length and 5 cm in diameter.

Assume isentropic flow in the nozzle and frictionless flow in the duct, with the air behaving as a perfect gas with constant specific heats.
10.3 The Rayleigh Line

• The Fanno continuity & energy eqs. and the Rayleigh continuity & momentum eqs are exactly the same as the continuity, momentum & energy eqs developed for the normal shock conditions.

• The locus of states before and after a normal shock appears on both the Rayleigh & Fanno lines.

• This figure displays Rayleigh & Fanno lines for the case of heat addition and area change for the same mass flow rate.

10.4 Normal Shock on the Rayleigh & Fanno Line: TS Diagram
10.4 Normal Shock on the Rayleigh & Fanno Line: TS Diagram

- The intersections of the curves at points 1 & 2 represent the flow ahead of and behind a normal shock.
- The increase in entropy for the shock is brought about by irreversibility's such as heat conduction and viscous dissipation occurring internal to the wave.
- An entropy increase for Fanno flow is brought about by friction.
- An entropy increase for Rayleigh flow is brought about by heat addition.
- The dashed line between 1 and 2 represents the fluid inside the shock, which is not in thermodynamic equilibrium.

10.5 Flow with Heat Addition and Area Change

- Such a flow can be found in the exhaust of gases through a rocket nozzle.
- Assume a steady, 1D frictionless flow and derive an expression relating the M# variation to changes in area & heat addition or loss.
- The continuity, momentum & energy eqs are:

\[
\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \\
\frac{dp}{\rho} + \frac{dV}{v} = 0 \\
dq = c_p dT_i
\]
10.5 Flow with Heat Addition and Area Change

- The perfect gas law in differential form, \( \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \) (10.23)

- Divide eq10.21 by \( p \) & substitute in eq 10.23, therefore

\[ \frac{d\rho}{\rho} + \frac{dT}{T} + \frac{\rho}{p} VdV = 0 \]

- Substitute eq 10.20,

\[ \frac{dA}{A} + \frac{dV}{V} = 0 \]

- Rewrite \( dV/V \) using \( M = V/a \)

\[ \frac{dV}{V} = \frac{dM}{M} + \frac{\gamma}{2} \frac{dT}{T} \]

- Therefore:

\[ -\frac{dA}{A} - \frac{dM}{M} + \frac{\gamma}{2} \frac{dT}{T} + \frac{\gamma M^2}{V} \frac{dV}{V} = 0 \]

\[ dq = c_p dT \]

10.5 Flow with Heat Addition and Area Change

\[ -\frac{dA}{A} - \frac{dM}{M} + \frac{\gamma}{2} \frac{dT}{T} + \gamma M^2 \left( \frac{dM}{M} + \frac{\gamma}{2} \frac{dT}{T} \right) = 0 \]

\[ -\frac{dA}{A} + \left( \frac{\gamma M^2}{2} - 1 \right) \frac{dM}{M} + \left( \frac{\gamma M^2}{2} + 1 \right) \frac{dT}{T} = 0 \] (10.24)

- Now to obtain a relationship for \( dT/T \), note the following, where

\[ T_i = T \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \]

for a perfect gas with constant specific heats. Now differentiate \( dT \), and

\[ \frac{dT}{T} = \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \frac{dT}{T} + \left( \gamma - 1 \right) M^2 \frac{dM}{M} \]
10.5 Flow with Heat Addition and Area Change

Now solving for $dT/T$ one obtains;

$$\frac{dT}{T} = \frac{dq}{c_p T} - \frac{M^2 (\gamma - 1)}{1 + \frac{\gamma - 1}{2} M^2} \frac{dM}{M}$$

• Then substitute into eq 10.24

$$- \frac{dA}{A} + \frac{\left(\gamma M^2 + 1\right) dq}{c_p T} \frac{1}{2 \left(1 + \frac{\gamma - 1}{2} M^2\right)} + \frac{\left(M^2 - 1\right) dM}{M} = 0$$

(10.25)

Notes:
• Except for special cases, direct integration of eq10.25 is not possible. However, the following general conclusions can be drawn:

i) $dq = 0$; eq 10.25 reduces to the isentropic flow eq., for $M = 1$ at $dA = 0$, which corresponds to the minimum area in the throat of a C-D geometry.

ii) With heat addition, the 2nd term in eq 10.25 is "+", so when $M = 1$ occurs, $dA/A$ is "+", which represents the diverging portion of the nozzle.

iii) Conversely, if heat is rejected from a gas stream (e.g., hot gas flow through a rocket nozzle), the sonic point occurs in the converging portion of the nozzle.
10.6 Flow with Friction and Heat Addition

• Consider flow in a constant area duct with heat transfer and friction.

\[
\frac{d\rho}{\rho} + \frac{dV}{V} = 0
\]  
(10.26)

• Therefore Conservation of Mass (\(dA = 0\))

\[
\frac{dp}{p} + \frac{dV}{V} = 0
\]

10.6 Flow with Friction and Heat Addition

• Conservation of Momentum

\[
dp + \frac{1}{2} \rho V^2 \frac{fdx}{D} + \rho V dV = 0
\]

\[
V = M \sqrt{\gamma RT} \implies \frac{dV}{V} = \frac{dM}{M} + \frac{1}{2} \frac{dT}{T}
\]

then substitute into the momentum eq and divide by \(p\),

\[
\frac{dp}{p} + \frac{1}{2} \gamma M^2 f \frac{dx}{D} + \gamma M^2 \frac{dM}{M} + \frac{1}{2} \gamma M^2 \frac{dT}{T} = 0
\]  
(10.27)

• Conservation of energy (no external work except friction)

\[
dq = c_p dT_t
\]
and we know

\[ T_i = T \left(1 + \gamma - \frac{1}{2} M^2 \right) \]

Therefore,

\[ \frac{dq}{c_p T} = \left(1 + \gamma - \frac{1}{2} M^2 \right) \frac{dT}{T} + (\gamma - 1) MdM \]

(10.28)

• For a perfect gas:

\[ \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \]

• Combining with eq 10.26 one obtains,

\[ \frac{dp}{p} = - \frac{dV}{V} + \frac{dT}{T} \]

• Using the definition of M #,

\[ \frac{dp}{p} = - \frac{dM}{M} + \frac{1}{2} \frac{dT}{T} \]

10.6 Flow with Friction and Heat Addition

• Substitute the above expression for dp/p into eq 10.27 along with dT/T from eq 10.28 and obtain;

\[ \frac{dq}{c_p T_i} \left[ \frac{1}{2} \left(1 + \gamma M^2 \right) \right] + \frac{1}{2} \gamma M^2 \frac{fdx}{D} \]

\[ = \left[ \frac{1 - \gamma M^2}{M} + \frac{(\gamma - 1)(1 + \gamma M^2) \frac{1}{2} M}{1 + \frac{\gamma - 1}{2} M^2} \right] dM \]

(10.29)

• Just as eq 10.25 indicates the effects of area change and heat addition on M #, eq 10.29 shows the effects of heat addition and friction on M#.