1.6 Conservation of Mass

- If \( m \) equals the total mass of the system, for a fixed mass system,

\[
\frac{dm}{dt} = 0
\]

- Recall \( x \) is an intensive property; \( x = \text{property/mass} \),

\[
0 = \frac{\partial}{\partial t} \iiint_{\forall} \frac{m}{m} \rho d\forall + \iint_{S} \frac{m}{m} \rho \vec{V} \cdot d\vec{A}
\]  

\( 1^\text{st} \) term \( \Delta \) the rate of change of mass within \( \forall \),

\( 2^\text{nd} \) term \( \Delta \) the net mass crossing \( S \), the boundary of the \( \forall \).

Note: By convention term 2 in (1.8) \( \Delta \) is positive for efflux and negative for influx
Example 1.3:
Air with a mass flow rate of 10 kg/s enters a rigid tank whose volume is 100 m³. Simultaneously, air exits the tank @ 2 kg/s. The temperature of the air inside the tank remains constant @ 300 K. Treating the air as a ideal gas find the rate of pressure rise inside the tank. The gas constant is, $R_{\text{air}} = 0.287 \text{kJ/kgK}$

Solution:
Example 1.4:
Two kg/s of liquid hydrogen (H₂) and 8 kg/s of liquid oxygen (O₂) are injected into a rocket combustion chamber. The gaseous products of combustion are expelled at high velocity through an exit nozzle. The nozzle exit diameter is 30 cm, and the density of the gases at the exit plane is 0.18 kg/m³.

Assuming steady, uniform conditions at the nozzle exhaust plane determine the exit velocity of the combustion products.
1.6 Conservation of Mass

2.2 Conservation of Mass in Differential Form

- The conservation of mass (also referred to as the continuity eq) states that mass cannot be created or destroyed (neglecting nuclear effects).
- Let’s now derive the conservation of mass by examining a small elemental fluid volume, $dV$, fixed in space.

Assume:
1) an elemental volume, $dV$,
2) No flow in Z-direction, 2D (consider unit width in Z-direction)

Sketch shows velocities and densities for mass flow balance through a fixed volume element in two dimensions.

**Equations**

\[
\rho \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial}{\partial x} (\rho u) dx + \rho v \frac{\partial v}{\partial y} + \frac{1}{2} \frac{\partial}{\partial y} (\rho v) dy = - \frac{\partial P}{\partial x} \]
1.6 Conservation of Mass

simplifying, the conservation of mass in 2-D becomes,
\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \]

• If one accounted for all three directions, the continuity equation becomes:
\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0 \]  
\hspace{2cm} (2.1a)

or in vector form
\[ \rho \nabla \cdot \mathbf{V} = \rho \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla \rho \]  
\hspace{2cm} (2.1b)

• Using the vector identity:
\[ \nabla \cdot (\rho \mathbf{V}) = \rho \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla \rho \]
\[ \text{DIV} (\rho \mathbf{V}) = \rho \text{Div} (\mathbf{V}) + \mathbf{V} \cdot \text{Grad} (\rho) \]

and substituting this identity into Eq (2.1b),
\[ \frac{\partial \rho}{\partial t} + \rho \text{Div} (\mathbf{V}) + \mathbf{V} \cdot \text{Grad} (\rho) = 0 \]

• Expanding this equation,
\[ \frac{\partial \rho}{\partial t} + \rho \left\{ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right\} + \left\{ u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \right\} = 0 \]  
\hspace{2cm} (2.1c)

• Recalling in Cartesian coordinates;
\[ \mathbf{V} = u \mathbf{i} + v \mathbf{j} + w \mathbf{k} \quad \text{and} \quad \nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \]

• Recalling the definition for the total or material derivative;
\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \]  
\hspace{2cm} (2.2)
1.6 Conservation of Mass

therefore,

\[
\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \quad (2.2a)
\]

• Substitute (2.2a) into (2.1c)

\[
\frac{D\rho}{Dt} + \rho \left\{ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right\} = 0 \quad (2.3a)
\]

\[
\frac{D\rho}{Dt} + \rho \text{DIV} \vec{V} = 0 \quad (2.3b)
\]

This is the conservation of mass, also referred to as the continuity equation.

Example 2.1: Air flows in a channel whose velocity at three adjacent points A, B, and C, which are 10 cm apart, are shown in the following figure.

Find the density gradient at point B if the fluid temperature is 40 degrees C and the absolute static pressure is 350 kPa.
1.6 Conservation of Mass

- Generalizing the conservation of mass for an arbitrarily shaped fixed volume.

Let: \( \vec{V} \) = velocity vector
\( \hat{n} \) = outward pointing unit vector, normal to the surface
\( dS \) = a surface element on the surface \( S \) enclosing the \( C \)

- The mass flow/unit time crossing \( S \) is
\[
\dot{m} = \int \rho \vec{V} \cdot \hat{n} \ dS
\]

- The rate of mass decrease within the control volume
\[
- \int_{C} \frac{\partial \rho}{\partial t} \ d\mathcal{V}
\]

CONSERVATION OF MASS: INTEGRAL FORM

\[
\frac{d}{dt} \iiint_{C} \rho \ d\mathcal{V} + \iiint_{S} \rho \left( \vec{V} \cdot \hat{n} \right) dS = 0
\]

- Using Gauss's theorem, the area integral can be transformed into a volume integral;
\[
\int_{S} \vec{A} \cdot \hat{n} dS = \int_{C} \nabla \cdot \vec{A} d\mathcal{V}
\]

therefore, one can rewrite the integral form of the conservation of mass,
\[
\int_{C} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho \vec{V} \right) \right] d\mathcal{V} = 0
\]

where the first term represents the time rate of change of \( \rho \) in the \( C \) and the 2nd term is the net mass flow/unit volume.
1.6 Conservation of Mass

\[ \int \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] \, d\mathcal{V} = 0 \]

where the first term represents the time rate of change of \( \rho \) in the \( \mathcal{V} \) and the 2nd term is the net mass flow/unit volume.

• Since this relation must hold for all arbitrary control volumes, the conservation of mass reduces to

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \quad (2.1b) \]

• In Cartesian coordinates:

\[ \vec{V} = u \hat{i} + v \hat{j} + w \hat{k} \quad \text{and} \quad \nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \]

expanding \( \nabla \cdot (\rho \vec{V}) \)

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0 \]

or

\[ \frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial z} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0 \]
1.6 Conservation of Mass

\[ \frac{\partial \rho}{\partial t} + \rho \text{Div} \left( \vec{V} \right) + \vec{V} \cdot \text{Grad} \rho = 0 \quad (2.1d) \]

• Using the definition for the total or substantial derivative of a quantity \( A \), which can operate on either a scalar (i.e., \( u, p, T, \rho \), etc.), or a vector (i.e., \( \vec{V}, m, \vec{V}, \text{etc.} \)),

\[
\frac{DA}{Dt} = \frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} + v \frac{\partial A}{\partial y} + w \frac{\partial A}{\partial z} \\
\frac{DA}{Dt} = \frac{\partial A}{\partial t} + \vec{V} \cdot \text{Grad} A
\]

therefore Eq 2.1d becomes,

\[
\frac{D\rho}{Dt} + \rho \text{Div} \left( \vec{V} \right) = 0 \quad (2.3)
\]

1.6 Conservation of Mass

\[
\frac{1}{\rho} \frac{D\rho}{Dt} = -\text{Div} \left( \vec{V} \right) \quad (2.3)
\]

• The above equation relates the total rate of change of density to

\[-\text{Div} \left( \vec{V} \right) \]

• For an incompressible flow, both;

\[
\frac{1}{\rho} \frac{D\rho}{Dt} = 0
\]

and

\[
\text{Div} \left( \vec{V} \right) = 0
\]