Turbulence Scales: When & Why

• The rate at which chemical reactions proceed depends upon the rate of mixing that takes place between the reactants, (i.e., the fuel and oxidizer).
• This intimate mixing is determined by the frequency and the size of turbulent motions.
• Droplet breakup also depends upon the size and energy of turbulent motions.
• Eddies that are too large tend to transport the drops but not necessarily breakup them up.
• Coalescence of drops depend upon the collision speed and frequency of the turbulent eddies.
• These parameters are correlated with the rate of energy dissipation, $\varepsilon$, and the length microscale related to the drop radius.
Small Scales in Turbulence

- As previously mentioned, viscosity can be very effective in smoothing out velocity fluctuations at the very small scales.

- The generation of small scale motions (i.e., fluctuations) is due to inertial interactions transmitted by the nonlinear terms in the equation of motion,
  \[
  \left( u_j \frac{\partial u_i}{\partial x_j} \right)
  \]

- The viscous terms prevent the generation of infinitely small scales of motion by dissipating small scale energy into heat.

- Since small scale motions have small time scales, one can therefore assume, that these motions are statistically independent of the relatively slow, large scale turbulent motions and the mean flow.

Small Scales in Turbulence

- Small scale motions depend only on the rate at which they are supplied energy by the large scales and the rate energy is dissipated due to kinematic viscosity.

- One can equate the rate of energy \( \overline{u^2} \) supplied to the small scale motions to the rate at which energy is dissipated.
  \[
  \left( \frac{d\overline{u^2}}{dt} \right) = \epsilon
  \]  (1.21).

- The rate of change of energy in the small scales is equal to the rate at which energy is dissipated into heat, raising the fluids internal energy.

- This is the basis for what is called Kolmogorov's Universal Equilibrium Theory of small scale structures.
Small Scales in Turbulence

• These small scales are referred to as Kolmogorov's Microscales of;

  i) Length: \( \eta = \left( \frac{\nu}{\epsilon} \right)^{1/4} \) (1.22a)
  ii) Time: \( \tau = \left( \frac{\nu}{\epsilon} \right)^{1/2} \) (1.22b)
  iii) Velocity: \( \upsilon = \left( \frac{\nu \epsilon}{\nu} \right)^{1/4} \) (1.22c)

• Where the microscale Reynolds number is:

\[
\text{Re}_\eta = \frac{\nu \eta}{\nu}
\]

• When the microscale (i.e., Kolmogorov's microscales) Reynolds number = 1, then the viscous forces acting on the small scales are as important as the non-linear inertia forces which produced the eddies.

Note:

a. On average the Kolmogorov microscale is statistically the smallest scale of motion present in the flow.

b. Breakdown of large scale motions is not due to viscous forces, but due to the inertial interactions and the resulting forces (represented by the non-linear term),

\[
\begin{pmatrix}
  u_j \\
  \frac{\partial u_i}{\partial x_j}
\end{pmatrix}
\]

Recall the large eddies;

i) do most of the transport of mass, momentum and energy

ii) these eddies can be as large as the width of the flow.
Summarizing length Scales

- Integral length scale:
  (large scale motions) $\sim (k^{3/2}/\nu)$

- Taylor Microscale:
  (length scale as turbulence moves to the isotropic state)
  $\sim (15\nu u'^2/\nu)^{1/2}$

- Kolmogorov length scale:
  (length scale of the smallest eddies) $\sim (\nu^{3/2}/\nu)^{1/4}$

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Summarizing Turbulent Scales

- Kinetic energy of the turbulent fluctuations is the sum of energies of turbulent eddies of different sizes,

$$k = \frac{1}{2} u_i u_i = \int_0^\infty E(\kappa) d\kappa$$

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Note:

i) Typical frequency ($f$) of the small scales is $\sim 10$ kHz,

ii) Kolmogorov scale typically ranges between $\eta \sim 0.01$ up to $0.1$ mm,

iii) Kolmogorov scale $\eta$ decreases with the increasing Re
Summarizing Turbulent Scales

\[ k = \frac{1}{2} \frac{u_i u_i}{m^2} \]  

kinetic energy of the turbulence

\[ \frac{1}{2} \frac{u_i u_i}{m^2} = \int_0^\infty E(\kappa) \]  
spectral energy is independent of viscosity in the inertial range

\[ \frac{d\kappa}{[1/m]} \]  

wave number

\[ = 2\pi/\lambda \]  

\[ \kappa = 2\pi/\lambda \]  

\[ \frac{d\kappa}{[1/m]} \]  

wave number

\[ E(\kappa) = \text{Const} * \kappa^{-5/3} \]  

Note:
- Supported by experiment and dimensional analysis, within the inertial subrange the spectral energy is related to:

\[ E(\kappa) = \text{Const} * \kappa^{-5/3} \]  

Note:
- 10 cm dia air jet (Champagne, 1978)
- Assumptions: turbulence is isotropic, incompressible, and stationary in time.

An Inviscid Estimate for the Dissipation Rate (\( \varepsilon \))

- The amount of kinetic energy/unit mass in the large scale turbulence is proportional to \( u^2 \).

- The rate of transfer is assumed to be proportional to \( (u/\ell) \); where \( \ell \) is the size of the large eddies or width of the flow and \( u \) is characteristic of the velocity fluctuations.

- Therefore, the rate of energy supplied to the small-scale eddies, (that is, rate of energy transferred by the large scales through inertial interactions) is;

\[ \left( \frac{d\bar{u}^2}{dt} \right) \sim \frac{u^2}{\ell/u} = \frac{u^3}{\ell} \simeq \frac{d\bar{u}^2}{dt} \left[ \text{Inertial} \right] \]  

Note: Molecular viscosity is not present.
An Inviscid Estimate for the Dissipation Rate ($\varepsilon$)

- As indicated earlier in Eq (1.21),
  \[
  \frac{du^2}{dt} = \varepsilon_{\text{Inertial}}
  \]

- so replacing the left hand side
  \[
  \varepsilon_{\text{Inertial}} = \frac{u^3}{\ell} \tag{1.24}
  \]

- Hence, viscous dissipation of energy can be estimated from the large scale dynamics (which is independent of viscosity).

- Even more interesting is the fact that energy dissipation proceeds at a rate governed by the inviscid inertial behavior of the large scale turbulent motions.

An Inviscid Estimate for the Dissipation Rate ($\varepsilon$)

- Recall, the large eddies lose a negligible fraction of their energy by direct viscous dissipation.

- The time scale associated with the large eddy viscous decay (or the rate of viscous decay of the large scale eddy) is $v/\ell^2$

- Therefore, large scale eddy energy losses due to viscosity proceeds at a rate of,

  \[
  \text{(1.25)}
  \]

  which is small compared to $(u^3/\ell)$, if the turbulent Re is large.

  \[
  Re_{\ell} = \frac{u \ell}{v}
  \]
An Inviscid Estimate for the Dissipation Rate ($\varepsilon$)

- The inertial mechanism is dissipative in that it creates smaller and smaller eddies, until the eddy size becomes so small that direct viscous dissipation of their K.E. is immediate.

Turbulence Scale Relations

1) The viscous time scale of the large eddies,

$$t_v = \frac{\ell^2}{\nu}$$

2) So the rate at which viscosity dissipates (or removes) energy from the large scale eddies (Eq 1.25),

3) Rate at which energy is supplied to the small scale motions, by the large scale motions, through inertial interactions and then dissipated (Eq 1.24) is,

$$\varepsilon_{inertial} = \frac{\bar{du}^2}{dt} \bigg|_{inertial} \approx \frac{u^2}{\ell/u} = \frac{u^3}{\ell}$$
Turbulence Scale Relations

• Forming a ratio of scales by substituting Eq (1.24) for $\varepsilon$, into the Kolomogorov microscales;

i) length ratio:

$$\frac{\eta}{\ell} \sim \left(\frac{\nu}{\varepsilon}\right)^{1/4}$$

which is the length ratio of small scale to large scale eddies

Rearranging;

Turbulence Scale Relations

ii) time ratio:

$$\frac{\tau}{t} \sim \left(\frac{\nu}{\varepsilon}\right)^{1/2} \ell / u$$

$$\frac{\tau}{t} \sim \left(\frac{\nu}{\varepsilon}\right)^{1/2} \frac{u}{\ell}$$

$$\frac{\tau}{t} \sim \left(\frac{\nu}{u^3/\ell}\right)^{1/2} \frac{u}{\ell}$$

$$\frac{\tau}{t} = \left(\frac{\nu u^2 \ell}{u^3 \ell^2}\right)^{1/2}$$
Turbulence Scale Relations

ii) time ratio (con't):

\[
\frac{\tau}{t} = \left( \frac{\nu}{u\ell} \right)^{1/2}
\]  

(1.27)

\[
\frac{\tau}{t} = \left( \frac{1}{Re} \right)^{1/2}
\]

which is the temporal ratio of small scale eddies to large scale eddies.

Note: As the turbulent Re increases, the Kolmogorov time scale \( \tau \) decreases with \( t (= \ell/u) \) remaining constant.

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Turbulence Scale Relations

iii) velocity ratio:

\[
\frac{\nu}{u} = \left( \frac{\nu \varepsilon}{u^3} \right)^{1/4}
\]

(1.28)

- These relations indicate that the length, time and velocity scales of the smallest eddies (Kolmogorov Microscales) are very much less than those of the largest eddies.
b) The main difference between two turbulent jet flows having different Re, but same overall length scale, is the size of the smaller eddies.
   i. Low Re, a "coarse" small scale structure is observed (fig a).
   ii. High Re, a relatively "fine" small scale motion is found (fig b).

\[
\frac{\tau}{t} \approx \left(\frac{1}{Re}\right)^{1/2}
\]

\[
\tau \sim \frac{\ell}{u} \text{Re}^{-1/2}
\]

- Since the vorticity \( (\omega) \) is the reciprocal of time \( (\omega = 1/\tau) \)

- Therefore, the vorticity of the small scale eddies are very much > that of the large scale motions.
Turbulence Scale Relations

d) From the velocity ratio (Eq1.28), energy in the small scale motions can be expected to be < the energy in the large scale eddies.

- Typical of turbulence, most of the energy is associated with the larger scales of motion, while the higher levels of vorticity are associated with the smaller scales of motions.

![Scaling relations on log-log paper](image)

Molecular and Turbulent Scales

- The Kolmogorov length and time scales are the smallest scales occurring in turbulent motion.

  i) Length: \[ \eta = (\nu^3/\varepsilon)^{1/4} \] (1.22a)
  
  ii) Time: \[ t = (\nu/\varepsilon)^{1/2} \] (1.22b)
  
  iii) Velocity: \[ \upsilon = (\nu^2/\varepsilon)^{1/4} \] (1.22c)

- The length and time scales decrease with increasing dissipation rates (higher fluctuating velocity).

  \[ \varepsilon \sim \frac{u^3}{\ell} \]

- Now, verify the continuum assumption as these scales decrease.

- First let us define the Knudsen number, \( Kn \), as the ratio of the molecular mean free path, \( \xi \) (i.e., molecular length scale) to the characteristic length scale (\( L \)) of the flow,

  \[ Kn = \frac{\xi}{L} \] (1.30)
Molecular and Turbulent Scales

- If instead we use $\eta$, the Kolmogorov length scale, one can define what is sometimes called The Microstructure Knudsen number,

$$\text{Kn}_T = \frac{\xi}{\eta} \quad (1.31a)$$

so,

$$\text{Kn}_T = \frac{\xi}{(\nu^3/\epsilon)^{1/4}}$$

- From the kinetic theory of gases; the kinematic viscosity can be approximated by,

$$\nu \sim \xi a$$

where "a" represents the speed of sound in the gas, which is approximately equal to the velocity scale of the molecular motion. So,

Define a turbulent Mach number ($M_T$) as the ratio of the turbulent velocity fluctuations ($u$) to the speed of sound.

- If the turbulent $M_T \sim 0(1)$ and $Re >> 1$ (as in most turbulent flows) then $\xi/\eta$ is small.

- When the $\text{Kn}_T < 0.01$, a pure continuum is said to exist.
Molecular and Turbulent Scales

Quizz/Example - Determine if the continuum assumption is valid for the following supersonic turbulent boundary layer with a 10% turbulent intensity (T.I.), a freestream $M = 2.5$, and the $T = 230^\circ$K. Let the boundary layer have a thickness $\delta = 2.5$ cm and the local speed of sound, $a = 305$ m/s.

Solution