CHAPTER 4
Kinetic Energy Equations

Mean Flow Kinetic Energy

• Momentum equation;

\[
U_j \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{T_{ij}}{\rho} \right)
\]

(3.1)

• Continuity equation

\[
\frac{\partial U_i}{\partial x_i} = 0
\]

(2.9a)

• Multiply momentum equation by \(U_i\) and rearrange each term

\[
U_i U_j \frac{\partial U_i}{\partial x_j} = U_i \frac{\partial}{\partial x_j} \left( \frac{T_{ij}}{\rho} \right)
\]

\[\text{1a} \quad \leftrightarrow \quad \text{2a}\]
Mean Flow Kinetic Energy

Term 1a) \[ U_i U_j \frac{\partial U_i}{\partial x_j} = U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} U_i U_j \right) \] (3.2a)

Term 2a) \[ \frac{1}{\rho} U_i \frac{\partial}{\partial x_j} (T_{ij}) = \frac{1}{\rho} \left[ \frac{\partial}{\partial x_j} (T_{ij} U_i) - T_{ij} \frac{\partial U_i}{\partial x_j} \right] \] (3.2b)

• Recall the "total" mean stress tensor, \( T_{ij} \) is symmetric

\[ T_{ij} = -P \delta_{ij} + 2 \mu S_{ij} - \rho u_i u_j \]

and the mean strain rate is defined by

\[ S_{ij} \equiv \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \]

Note: [symmetric tensor] * [anti-symmetric tensor] / 0

Mean Flow Kinetic Energy

• Rewriting the deformation rate

\[ \frac{\partial U_i}{\partial x_j} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{1}{2} \left( \frac{\partial U_j}{\partial x_i} - \frac{\partial U_i}{\partial x_j} \right) \]

\[ \leftrightarrow S_{ij} \quad (\text{symmetric}) \quad a_{ij} = a_{ji} \]

\[ \leftrightarrow \text{RotationTensor} \quad (\text{anti-symmetric}) \quad a_{ij} = -a_{ji} \]

• So the 2nd term on the right hand side simplifies to:

\[ T_{ij} \frac{\partial U_i}{\partial x_j} = T_{ij} S_{ij} - T_{ij} \left[ \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right) \right] \]

\[ \leftrightarrow (\text{sym,anti-sym.}) = 0 \]

• Applying item a, the above term simplifies to

\[ T_{ij} \frac{\partial U_i}{\partial x_j} = T_{ij} S_{ij} \]
Mean Flow Kinetic Energy

- So Eq (3.2b) can be rewritten as

\[
\frac{1}{\rho} U_j \frac{\partial}{\partial x_j} (T_{ij}) = \frac{1}{\rho} \left[ \frac{\partial}{\partial x_j} (T_{ij} U_i) - T_{ij} S_{ij} \right] \quad (3.2.c)
\]

- Equate (3.2a) to (3.2c) to obtain the mean flow kinetic energy eq;

\[
\rho U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} U_i U_j \right) = \frac{\partial}{\partial x_j} \left( T_{ij} U_i \right) - T_{ij} S_{ij}
\]

Mean Flow KE \hspace{1cm} Spatial Transport of Mean Flow KE by the Total Mean Stress \hspace{1cm} Deformation Work

Total rate of Change of KE in the mean flow

1 \hspace{1cm} 2

Notes:
- a. Recall the total derivative of the quantity “A” is;

\[
\frac{DA}{Dt} = \frac{\partial A}{\partial t} + U_j \frac{\partial A}{\partial x_j}
\]

Mean Flow Kinetic Energy

b. Term "1" integrates to "0" if the integration refers to a CV on whose surface either \(T_{ij}\) or \(U_i\) vanishes. Recall the divergence theorem;

\[
\int \frac{\partial}{\partial x_j} (T_{ij} U_i) d\mathcal{V} = \int_{S} T_{ij} U_i n_j ds
\]

Therefore, Term "1" can only redistribute the energy within the control volume, while, Term "2", the Deformation work term, \(T_{ij} S_{ij}\) can change the total amount of mean flow K.E.

c. The Deformation Work term \((T_{ij} S_{ij})\) represents the mean flow K.E. that is lost to (or retrieved from) the inertial or viscous agency that generates the stress.

For example, viscous shear stresses perform deformation work on the mean flow which increases the internal energy of the fluid at the expense of the turbulent kinetic energy.

This term is non-zero upon integration around the CV and accounts for the transport of energy into and out of the system.
Mean Flow Kinetic Energy

HW or Quiz Example: A Pure Shear Flow (Couette flow)

Assume:

i. variables depend on $x_2$ only
ii. $U_2 = 0$
iii. Steady State

Determine the resulting mean flow KE eq for this flow in terms of viscous and turbulent stresses.

Solution: In Class Exercise
In summary, this example demonstrates that the turbulent and viscous stresses are deforming the mean flow field and the mean flow field must work against these stresses to maintain itself.

--- The End ---
Mean Flow Kinetic Energy

Note:

a. The contribution of pressure to deformation work in an incompressible flow is zero.

\[ -T_{ij}S_{ij} = +P\delta_{ij}S_{ij} - 2\mu S_{ij}S_{ij} + \rho \overline{u_iu_j}S_{ij} \]

Recall \( \delta_{ij} = 1 \); only when \( i = j \), and by continuity

\[ S_{ii} = \frac{\partial U_i}{\partial x_i} \equiv 0 \]

Therefore, the pressure does not contribute to bringing energy into or out of the mean flow in an incompressible fluid.

b. Deformation work is due to shear stresses (viscous and turbulent i.e., inertial).

c. Contribution of viscous stresses to the deformation work is always negative. Viscous deformation work always represents a loss in K.E. The quantity \( 2\mu S_{ij}S_{ij} \) is called the "viscous dissipation" term. (Dissipation is related to the strain rate, not the vorticity.)

d. The contribution of the Reynolds stresses to the deformation work in most flows is also dissipative; negative values of \( \overline{u_iu_j} \) tend to occur with \( '+' S_{ij} \).

(However, \( '+' \) values of \( \overline{u_iu_j}S_{ij} \) can also occur, but even then this region is only a small fraction of the entire flow.)

e. Since turbulent stresses perform deformation work on the mean flow, it is the K.E. of the turbulence (TKE) which benefits from this work.

Hence \textbf{turbulent energy production} is defined as

\[ -\rho \overline{u_iu_j}S_{ij} \]
Effects of Viscosity

\[ \rho U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} U_i U_i \right) = \frac{\partial}{\partial x_j} (T_{ij} U_i) - T_{ij} S_{ij} \]  

(3.3)

Term 1) represents the transport of mean flow K.E. by stress \( T_{ij} \).

Term 2) \( T_{ij} S_{ij} \), is called the deformation work.

- Recall that the spatial gradient terms always represent a transport of a conserved quantity from one place to another.

  This is verified by integrating such terms over the entire flow control volume and obtaining zero.

- Expand the mean flow K.E. Eq (3.3) by substituting the mean stress tensor;
Effects of Viscosity

- Term 3 is the transport of mean flow K.E. by Reynold's stresses
  \[ \frac{\partial}{\partial x_j} (-u_i u_j U_j) \]
- Term 4 is the loss of K.E. as a result of viscous deformation work (i.e. viscous dissipation)
  \[ -2\nu S_{ij} \]
- Term 5 is the deformation work produced by turbulent stresses (turbulent production usually < 0)
  \[ \overline{u_i u_j S_{ij}} \]

Recall if \( U_i T_{ij} = 0 \) on the surface of a control volume, these first 3 terms can only redistribute energy inside that control volume.

Now let's determine the importance of the viscous and Reynold's stresses (terms 2 and 3) in the transport term.

Effects of Viscosity

- Performing dimensional reasoning on term "3" of the mean flow K.E. equation, let \( u = u_c \) and "\( \ell \)" the length scale,
  \[ S_{ij} \approx \frac{\partial U_i}{\partial x_j} \sim \frac{u_c}{\ell} \]
  so term 3 can be estimated as
  \[ \overline{u_i u_j U_i} \approx u_c u_c U_i \]
  \[ \overline{u_i u_j U_i} \approx u_c (\ell S_{ij}) U_i \]

Taking the ratio of the kinetic energy transport by terms 3 & 2,

\[
\frac{\text{turbulent (Reynolds) Stress}}{\text{viscous stress}} \approx \frac{u_c (\ell S_{ij}) U_i}{2\nu U_j S_{ij}}
\]
\[
\frac{\text{turbulent (Reynolds) stress}}{\text{viscous stress}} \approx \frac{u_c \ell}{\nu}
\]
Effects of Viscosity

\[ \frac{\text{turbulent (Reynolds) stress}}{\text{viscous stress}} \approx \frac{u_c \ell}{\nu} \]

- Therefore, if the Re number is large, the effect of the viscous stresses on the transport on mean flow K.E. around the mean flow system (i.e., the redistribution) is negligible compared to the turbulent (i.e., Reynolds) stresses.

- Repeating the same non-dimensional procedure on term 5, the turbulent production becomes

\[ \overline{u_i u_j S_{ij}} \approx u_c \ell \overline{S_{ij}} \]

\[ \overline{u_i u_j S_{ij}} \approx u_c \ell \overline{S_{ij} S_{ij}} \]

- Compare the viscous and turbulent contributions to deformation work (terms 4 and 5).

So if the Re is large, the effect of viscosity on the total dissipation of mean flow K.E. is small in comparison to the turbulent processes.

Hence, the gross structure of turbulent flows tends to be virtually independent of viscosity.