7.0 Heat Transfer in an External Laminar Boundary Layer

7.1 Introduction

In this chapter, we will assume:

1) That the fluid properties are constant and unaffected by temperature variations.
2) The thermal & momentum boundary layers, which develop along a given surface, are not influenced by any other adjacent surface.
3) Body forces are negligible, & the velocity is sufficiently low such that the viscous dissipation term, in the energy eq, can be neglected.

7.2 Flow Over a Flat Plate at Constant Temperature

Let a fluid with \( U_\infty = \text{constant} \) flow over a constant temperature flat plate. A constant wall temperature (\( T_o = \text{constant} \)) indicates that the thermal and momentum boundary layers will be initiated at the same location (i.e., leading edge of the plate).

Define the non-dimensional temperature as;

\[
\theta^+ = \frac{T_o - T}{T_o - T_\infty}
\] (7.1)

The steady state energy eq for a gaseous boundary layer;

\[
\rho \mathcal{C}_p \frac{\partial T}{\partial x} + \rho \mathcal{C}_p \frac{\partial T}{\partial y} - \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) - \mu \left( \frac{\partial u}{\partial y} \right)^2 = 0
\] (7.2a)

If \( k = \text{constant} \), which is a good assumption for liquids and a reasonable assumption for gases with small temperature gradients, eq 7.2a becomes;

\[
u \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} - \frac{k}{\rho \mathcal{C}_p} \frac{\partial^2 T}{\partial y^2} - \mu \left( \frac{\partial u}{\partial y} \right)^2 = 0
\] (7.3a)
Recall the thermal diffusivity, \( \alpha = k/\rho c \) and \( \Pr = \frac{\nu}{\alpha} = \frac{\mu c}{k} \);

\[
\frac{u \partial T}{\partial x} + \frac{v \partial T}{\partial y} - \alpha \left[ \frac{\partial^2 T}{\partial y^2} + \frac{\Pr}{c} \left( \frac{\partial u}{\partial y} \right)^2 \right] = 0
\]

(7.3b)

Note the last term, which represents the dissipation function, depends not only on the velocity gradient, but also on the \( \Pr \).

- Viscous energy dissipation can be important for high Prandtl fluids (i.e., oils) even under moderate velocity gradients.

- For gases, with \( \Pr \approx 1 \), the velocity must approach the speed of sound before this term is significant.

- If viscous dissipation can be neglected;

\[
\frac{u \partial \theta^+}{\partial x} + \frac{v \partial \theta^+}{\partial y} - \alpha \frac{\partial^2 \theta^+}{\partial y^2} = 0
\]

(7.3c)

For the current problem the boundary conditions are

\[
\begin{align*}
\theta^+ &= 0 \quad @ \quad y = 0 \\
\theta^+ &= 1 \quad @ \quad y \to \infty \\
\theta^+ &= 1 \quad @ \quad x = 0
\end{align*}
\]

The hydrodynamic portion of this problem was solved using eqs 6.1 and 6.2, resulting in eq 6.18:

\[
c_f = \frac{1.328}{(\text{Re}_c)^{1/2}}
\]

- Eq 7.3c is very similar to the corresponding momentum eq 6.1. In fact, if the \( \alpha = \nu \) (i.e., \( \Pr = 1 \)) the same functional solution satisfies both equations & b.c's.
Therefore, for a fluid with Pr=1, the nondimensional velocity & temperature profiles are similar and grow along the plate at the same rate. The condition of similar V and T profiles leads to a relation between the Stanton number (St) and the local skin friction coefficient (c_f);

\[ St_x = \frac{h_x}{\rho C_p U_\infty} \quad \text{and} \quad St_x = \frac{c_f x}{2} \quad \text{and} \quad St = \frac{Nu}{Pe} = \frac{Nu}{Re Pr} \quad (7.4) \]

- Using a general similarity a solution to eq 7.3c can be obtained;

\[ \theta^+ = \theta^+ (\eta) \quad \text{where} \quad \eta = \frac{y}{\sqrt{v x / U_\infty}} \]

and

\[ u = \frac{\partial \psi}{\partial y} \quad \nu = -\frac{\partial \psi}{\partial x} \quad \psi = \sqrt{v x U_\infty} F \]

Substituting into eq 7.3c;

\[ \theta^{++} + \frac{Pr}{2} F \theta^+ = 0 \quad (7.5) \]

where both \( \theta^+ \) and F are functions of \( \eta \).

The F function was computed in chapter 6 from solution to the Blasius eq.

- Rewriting eq 7.5;

\[ \frac{d \theta^+}{d \eta} + \frac{Pr}{2} F \theta^+ = 0 \]

\[ \frac{d \theta^+}{\theta^+} + \frac{Pr}{2} F d \eta = 0 \]
\[ \theta^+ = C_1 \exp \left( -\frac{\Pr}{2} \int_{o}^{\eta} F \, d\eta \right) \]  
(7.6)

\[ \theta^+ = C_1 \int_{0}^{\eta} \exp \left( -\frac{\Pr}{2} \int_{o}^{\eta} F \, d\eta \right) d\eta + C_2 \]  
(7.7)

Applying the b.c., \( \theta^+ = 0 \) @ \( \eta = 0 \); gives \( C_2 = 0 \)

the second b.c., \( \theta^+ = 1 \) @ \( \eta \to \infty \); gives \( C_1 \)

\[ C_1 = \frac{1}{\int_{0}^{\infty} \left[ \exp \left( -\frac{\Pr}{2} \int_{o}^{\eta} F \, d\eta \right) \right] d\eta} \]  
(7.8)

\[ \theta^+(\eta) = \frac{\int_{0}^{\eta} \left[ \exp \left( -\frac{\Pr}{2} \int_{o}^{\eta} F \, d\eta \right) \right] d\eta}{\int_{0}^{\infty} \left[ \exp \left( -\frac{\Pr}{2} \int_{o}^{\eta} F \, d\eta \right) \right] d\eta} \]  
(7.9)

- Recall:

\[ h_x = \frac{q_o^\prime\prime}{T_o - T_\infty} \quad \text{and} \quad q_o^\prime\prime = -k \frac{\partial T}{\partial y} \bigg|_o = -k (T_\infty - T_o) \frac{\partial \theta^+}{\partial y} \bigg|_o \]
\[ \dot{q}'' = k (T_o - T_\infty) \left[ \frac{\partial \theta^+}{\partial \eta} \frac{\partial}{\partial \eta} \right]_o \]

\[ \dot{q}'' = \frac{k (T_o - T_\infty)}{\sqrt{v x U_\infty}} \theta^+ (0) \]

or in terms of Nux;

\[ Nux = \frac{h x}{k} = \frac{x}{\sqrt{v x U_\infty}} \theta^+ (0) \]

\[ Nux = \frac{\theta^+ (0)}{\sqrt{(v x U_\infty)}} = Re_x^{1/2} \theta^+ (0) \]

but from eq (7.6):

\[ \theta^+ (0) = C_1, \]

\[ Nux = \frac{Re_x^{1/2}}{\int_0^\infty \exp\left( -\frac{Pr}{2} \int_0^\infty F d\eta \right) d\eta} \quad (7.10) \]

- \( F(\eta) \) was evaluated during the solution to the momentum eq.(Chapt 6). Therefore, the integral in eq (7.10) can be evaluated for any Pr (see table below).

<table>
<thead>
<tr>
<th>Pr</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
<th>0.5</th>
<th>0.7</th>
<th>1.0</th>
<th>7.0</th>
<th>10.0</th>
<th>15.0</th>
<th>50</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Nux ) Re_x^{-1/2}</td>
<td>0.0173</td>
<td>0.0516</td>
<td>0.140</td>
<td>0.259</td>
<td>0.292</td>
<td>0.332</td>
<td>0.645</td>
<td>0.730</td>
<td>0.835</td>
<td>1.247</td>
<td>1.572</td>
<td>3.387</td>
</tr>
</tbody>
</table>

Note: In external flow problems the film temperature \( (T_f) \) is used to evaluate thermophysical properties.
For the Pr range: $0.5 < Pr < 15$

\[ Nu_x = 0.332 \, Pr^{1/3} \, Re_x^{1/2} \]  \hspace{1cm} (7.11)

Note:
1- This correlation is good for Pr up to about 1,000 to within 2%.
2- For Pr < 0.5 this error noticeability increases.
   a. When the Pr = 0.1 the error increases to 10%.
   b. Reducing the Pr to 0.01 the error increases to 39%.

The following temperature profiles are displayed as a f(η)

Temperature profiles for a laminar constant property boundary layer, with $U_\infty$ and $T_0$ constant

Note:
1. Dependence of thermal boundary layer thickness ($\Delta$) on Pr.
2. Low Pr $\Rightarrow$ $\Delta > \delta$
3. High Pr $\Rightarrow$ $\Delta < \delta$
4. $Pr = 1$ $\Rightarrow$ $\Delta = \delta$
• When $Pr << 1 (\Delta >> \delta)$;

Let’s assume $u = U_\infty$, a slug flow problem. Therefore, $F' = 1 = u/U_\infty$ and

$$Nu_x = 0.565 \Pr^{1/2} \Re^{1/2}_x$$

• If $Pr >> 1 (\Delta << \delta)$;

The thermal boundary layer is entirely within the momentum boundary layer where the velocity profile is linear. From chapter 6, table 8-1;

$$F''(0) = 0.3321$$

and the final solution;

$$Nu_x = 0.332 \Pr^{1/3} \Re^{1/2}_x$$

Presenting this solution in terms of the Stanton number,

$$St_x = \frac{h_x}{\rho C_p U_\infty} = \frac{Nu_x}{Pe} = \frac{Nu_x}{\Re_x \Pr}$$

$$St \Pr^{2/3} = 0.332 \Re^{-1/2}_x$$

(7.12)

$$St_x \Pr^{2/3} = \frac{c_{fx}}{2} \begin{cases} \text{Reynolds–Colburn} \\ \text{Analogy} \end{cases}$$

Note, $h$ decreases in the flow direction as the boundary layer becomes thicker, and as $x \to 0$, $h$ increases and becomes infinite @ $x=0$. This is due to an infinitely large temperature gradient at the surface, at $x = 0$.

• Inspection of eq 7.11 or 7.12 indicates, $h = c x^{-1/2}$
To calculate the total heat transfer rate, integrate \( h \) over the length of the plate (i.e., \( x \)) and obtain a mean \( h \),

\[
h_m = \frac{1}{x} \int_o^x h_x \, dx = \frac{c}{x} \int_o^x x^{-1/2} \, dx = 2h_x
\]

(7.13)

The mean Nusselt # (\( \text{Num} \))

\[
\text{Nu}_m = 0.664 \Pr^{1/3} \Re_x^{1/2}
\]

(7.14a)

or divide by the product of (\( \Re \) \( \Pr \))

\[
\text{St}_m \Pr^{2/3} = 0.664 \Re_x^{-1/2}
\]

(7.14b)

Eqs 7.12 & 14 are often referred to as the **Pohlhausen solutions**.

- Rewriting the Stanton number in terms of the enthalpy thickness (\( \Delta_2 \)), rather than the distance (\( x \)) along the plate.

\[
\text{St} \Pr^{4/3} = 0.2205/\Re_{\Delta_2}
\]

(7.15)

Where the \( \Delta_2 \) is computed from eq (7.14a) with \( i_{s,o} \) representing the freestream enthalpy w.r.t the wall (o) temperature,

\[
\Delta_2 = \int_o^\infty \frac{\rho u i_s}{(\rho u)_\infty i_{s,o}} \, dy
\]

(7.A1a)

For a low velocity, constant property flow of a perfect gas, \( i_s = c(T - T_\infty) \) and

\[
\Delta_2 = \int_o^\infty \frac{u}{U_\infty} \left[ \frac{T - T_\infty}{T_\infty - T_\infty} \right] \, dy
\]

(7.A1b)

To a good approximation, eq’s (7.A1) is also applicable to a constant property liquid flow.
7.2 Flow Along a Constant Temperature Plate with \( U_\infty = cx^m \)

Since the similarity solutions were obtained for the momentum eq, one may expect a similarity solution for the energy eq.

Using the same similarity variables as those used for the momentum eq, the energy eq is transformed to an ODE similar to eq 7.5, with the Pr in the previous solution (eq 7.9) replaced by Pr(m+1).

The solution for F is obtained from the wedge flow solutions (in terms of m). The following table lists the local \( \text{Nu}_x \) for various values of m & Pr,

\[ \text{Nu}_x \, \text{Re}_x^{-1/2} = \text{Constant} \]

<table>
<thead>
<tr>
<th>( m = \frac{x}{U_\infty} \frac{dU_\infty}{dx} )</th>
<th>( \text{Pr} )</th>
<th>0.7</th>
<th>0.8</th>
<th>1.0</th>
<th>5.0</th>
<th>10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decelerating Flow</td>
<td>0.0753</td>
<td>0.242</td>
<td>0.253</td>
<td>0.272</td>
<td>0.457</td>
<td>0.570</td>
</tr>
<tr>
<td>Flat plate</td>
<td>0</td>
<td>0.292</td>
<td>0.307</td>
<td>0.332</td>
<td>0.585</td>
<td>0.730</td>
</tr>
<tr>
<td>2D Stagnation Point Flow</td>
<td>0.111</td>
<td>0.331</td>
<td>0.348</td>
<td>0.378</td>
<td>0.669</td>
<td>0.851</td>
</tr>
<tr>
<td></td>
<td>0.333</td>
<td>0.384</td>
<td>0.403</td>
<td>0.440</td>
<td>0.792</td>
<td>1.013</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.496</td>
<td>0.523</td>
<td>0.570</td>
<td>1.043</td>
<td>1.344</td>
</tr>
<tr>
<td></td>
<td>4.0</td>
<td>0.813</td>
<td>0.858</td>
<td>0.938</td>
<td>1.736</td>
<td>2.236</td>
</tr>
</tbody>
</table>

- Recall, \( m \) represents the various types of wedge flow fields from which,

\[
\frac{h}{k} \left[ \frac{x U_\infty}{\nu} \right]^{-1/2} = \text{const.}
\]

Or

\[
h = \text{const} \, k \, x^{-1/2} \left[ U_\infty / \nu \right]^{1/2}
\]

- For wedge flow problems where \( U_\infty = Cx^m \) the heat transfer coefficient,

\[
h = \frac{\text{const} \, k}{\nu^{1/2}} \, x^{(m-1)/2}
\]

Note:
1- For \( m = 1 \), indicates a 2D stagnation point flow where \( h = \text{constant} \) and does not vary with \( x \).
a. therefore for $m = 1$ all temperature profiles are similar

b. a constant $h$ can only mean that the thermal boundary layer is of constant thickness.

2- For $m < 1$, including negative $m$ values and zero (flat plate case, $m=0$) $h$ starts infinitely large @ $x = 0$ and decreases along the plate.

3- For $m > 1$, $h$ starts @ zero and increases with $x$.

- Computing the mean conductance, $h_m$ which is defined from 0 to $x$ as,

\[
h_m = \frac{1}{x} \int_0^x h \, dx
\]

then,

\[
h_m = \frac{1}{x} \int_0^x \frac{Ck}{\nu^{1/2}} x^{(m-1)/2} \, dx
\]

\[
h_m = \frac{2}{m+1} \frac{Ck}{\nu^{1/2}} x^{(m-1)/2}
\]

Thus,

\[
\frac{h_m}{h} = \frac{2}{m+1}
\]

(7.16)

When $m = 1$ and $Pr \approx 1$;

\[
Nu_x = 0.57 \, Pr^{0.4} \, Re_x^{1/2}
\]

(7.17)

### 7.3 Flow Along a Constant Temperature Plate with Injection or Suction

In chapter 6 a similarity solution to the momentum boundary layer eq for the case $V_o \neq 0$ and $U_\infty = Cx^m$ was obtained.

The solution indicated that $V_o \propto x^{(m-1)/2}$

Using eq 7.5 as the energy eq and b.c’s, the solution, eqs 7.9 & 10 are valid.

- Recall for $m \neq 0$, the Pr is replaced by $Pr(m+1)$ as noted in a previous section on wedge flow solutions.
Therefore, it is only necessary to employ the appropriate values of $F(\eta)$ from the solutions to the momentum eq for the various values of the blowing parameter:

$$\frac{V_o}{U_\infty} \text{Re}_x^{1/2}$$

The following tables provide results for $\text{Nu}_x$, under various blowing rates, $\text{Pr}$, and $m$ values for laminar boundary layer flow, respectively.

Values of \{Nu$_x$ Re$_x^{-1/2}$\} for small range Pr and rates of blowing or suction, for a constant property laminar boundary layer on a flat plate ($m=0$); where $T_o$, $T_\infty$, $U_\infty$ are all constants.

<table>
<thead>
<tr>
<th>$\frac{V_o}{U_\infty} \text{Re}_x^{1/2}$</th>
<th>Pr = 0.7</th>
<th>Pr = 0.8</th>
<th>Pr = 1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.50</td>
<td>1.850</td>
<td>2.097</td>
<td>2.59</td>
</tr>
<tr>
<td>-0.750</td>
<td>0.772</td>
<td>0.797</td>
<td>0.945</td>
</tr>
<tr>
<td>-0.250</td>
<td>0.429</td>
<td>0.461</td>
<td>0.523</td>
</tr>
<tr>
<td>0</td>
<td>0.292</td>
<td>0.307</td>
<td>0.332</td>
</tr>
<tr>
<td>0.250</td>
<td>0.166</td>
<td>0.166</td>
<td>0.165</td>
</tr>
<tr>
<td>0.375</td>
<td>0.107</td>
<td>0.103</td>
<td>0.0937</td>
</tr>
<tr>
<td>0.500</td>
<td>0.0517</td>
<td>0.0458</td>
<td>0.0356</td>
</tr>
<tr>
<td>0.619</td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>
In general these tables indicate that blowing or injection causes a decrease in the rate of heat transfer, while suction results in increased heat transfer.

Injection should be applied when protection (i.e., cooling) of surfaces exposed to hot gases is desired.

Examples are turbine blades, rocket engine nozzles, and reentry vehicle nose cones.
Note:

i- Implicit in these solutions are the fact that the injected fluid is the same as the freestream fluid, and its temperature is equal to that of the surface it passes through.

ii- The same condition holds for the case of suction.

As injection is increased, the $\text{Nu} \rightarrow 0$ which means the boundary layer is completely blown off, & the fluid temperature gradient @ the surface is zero.

Injection of a fluid, different from the freestream fluid involves both the diffusion of heat and mass.

If the suction rate is increased to very large levels, the heat transfer rate becomes equal to the change in enthalpy of the fluid drawn in because the fluid approaches the surface temperature at the surface.

7.4 Flow Along a Flat Plate w/ Constant $U_\infty$ and an Unheated Starting Length

Apply the energy integral to a laminar incompressible boundary layer, growing on a flat plate with a zero pressure gradient.

Assume the heating does not start at the leading edge, $x = 0$ where the hydrodynamic boundary layer begins, but at a position $x = \xi$ downstream from the leading edge.
Assume:

1- \( \Delta < \delta \)

2- Cubic velocity profile; 
\[
\frac{u}{U_\infty} = \frac{3}{2} \frac{y}{\delta} - \frac{1}{2} \left( \frac{y}{\delta} \right)^3
\]

(7.24)

3- Cubic temperature profile; 
\[
\frac{\theta}{\theta_o} = \frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \left( \frac{y}{\Delta} \right)^3
\]

(7.25)

Where \( \theta = (T_o - T) \) and \( \theta_o = (T_o - T_\infty) \);

and \( \Theta^+ = \frac{\theta}{\theta_o} \)

Substitute eqs 7.24 & 25 into the enthalpy thickness relation (for constant properties) & integrate,

\[
\Delta_2 = \int_0^\infty \frac{u}{U_\infty} \left[ \frac{T - T_\infty}{T_o - T_\infty} \right] dy
\]

(6.15)

Note: Only need to integrate to \( \Delta \), thickness of the thermal boundary layer.

The integration yields,

\[
\Delta_2 = \frac{3}{20} \frac{\Delta^2}{\delta} - \frac{3}{280} \frac{\Delta^4}{\delta^3}
\]

(7.26)

The ratio of thermal to hydrodynamic boundary layer thicknesses is defined,

\[
r = \frac{\Delta}{\delta} \quad \text{or} \quad \Delta = r \delta
\]

(7.27)

Now substitute for \( \Delta \) in eq 7.26;
Differentiate \( \Delta_2 \) w.r.t. “x”

\[
\frac{d\Delta_2}{dx} = 3\delta \left( \frac{1}{10} r - \frac{1}{70} r^3 \right) \frac{dr}{dx} + 3 \left( \frac{1}{20} r^2 - \frac{1}{280} r^4 \right) \frac{d\delta}{dx}
\]

If \( r < 1 \), the second term within each of the parentheses are small, and

\[
\frac{d\Delta_2}{dx} = \frac{3\delta r}{10} \frac{dr}{dx} + \frac{3r^2}{20} \frac{d\delta}{dx}
\]

using an alternate form of the energy integral (eq 6.21)

\[
\frac{d\Delta_2}{dx} = \frac{\dot{q}''_o}{\rho U_\infty c (T_o - T_\infty)} \tag{7.28a}
\]

Expressing the heat flux, \( \dot{q}''_o \) in terms of temperature,

\[
\frac{d\Delta_2}{dx} = \frac{-k (\partial T/\partial y)_o}{\rho U_\infty c (T_o - T_\infty)} \tag{7.28b}
\]

Substituting this expression for \( d\Delta_2/dx \) and evaluating the gradient at the surface, using eq 7.25,

\[
2r^2 \delta^2 \frac{dr}{dx} + r^3 \delta \frac{d\delta}{dx} = \frac{10\alpha}{U_\infty}
\]

Recall, the solutions for \( \delta \) and \( \delta \frac{d\delta}{dx} \) are available from chapter 6 solutions of the hydrodynamic boundary layer,

\[
\delta = 4.64 \sqrt{v x / U_\infty} \tag{7.29}
\]
Substituting for these terms, the following ODE for “r” is obtained,

\[ r^3 + 4r^2 \frac{dr}{dx} x = \frac{13}{14} Pr^{-1} \]

or

\[ r^3 = cx^{-3/4} + \frac{13}{14} Pr^{-1} \]

Evaluating the constant c with the b.c., \( r = 0 \) at \( x = \xi \)

\[ r = \frac{Pr^{-1/3}}{1.026} \left[ 1 - \left( \frac{\xi}{x} \right)^{3/4} \right]^{1/3} \quad (7.30a) \]

For \( \xi = 0 \), heating begins at the leading edge, \( x = 0 \) (no unheated starting length).

\[ r = 0.975 Pr^{-1/3} \quad (7.30b) \]

When \( Pr = 1 \), the two boundary layers have the same thickness (\( \delta = \Delta \)).

When \( Pr > 1 \), \( \delta > \Delta \).

When \( Pr < 1 \), \( \delta < \Delta \);

However, this solution is not applicable and therefore not accurate, because it was assumed that \( \Delta < \delta \) in the development of these solutions. Thus, care must be exercised in using this solution for \( Pr < 1 \). Yet, for gases in the range \( 0.5 < Pr < 1 \) reasonable results are possible.

- The heat transfer rate at the surface, using the assumed temperature profile,

\[ \dot{q}_o'' = -k \left[ \frac{\partial T}{\partial y} \right]_o = h \theta_o \]

Substituting eqs 7.25, 29, and 30a for \( \theta, \delta, \) and \( r, \) and then solving for \( h, \)
Determining the local $\text{Nu}_x$,
\[
\text{Nu}_x = \frac{0.332 \Pr^{1/3} \text{Re}_x^{1/2}}{1 - (\xi / x)^{3/4}}^{1/3}
\]

For the case of no unheated starting length ($\xi = 0$);
\[
\text{Nu}_x = 0.332 \Pr^{1/3} \text{Re}_x^{1/2}
\]

which is identical to the exact solution, eq7.11.

### 7.5 Flow Along a Flat Plate with a Constant $U_\infty$ and an Arbitrary Specified Constant Surface Temperature

The following section will examine the heat transfer for a constant property, constant $U_\infty$ fluid passing over a flat plate with a step change in the boundary condition, that is $T_o = T_\infty$ for $x < \xi$ and $T_o = \text{constant (and} \neq T_\infty\text{)}$ for $x > \xi$.

Solution to eq 7.3c follows, for an arbitrary variation in surface temperature, $T_o$ for a constant $U_\infty$ and $T_\infty$;

\[
\theta^+(\xi, x, y) = \frac{T_o - T}{T_o - T_\infty}
\]

\[
T - T_\infty = \int_{0}^{x} \left[1 - \theta^+(\xi, x, y)\right] \frac{dT_o}{d\xi} d\xi + \sum_{i=1}^{k} \left[1 - \theta^+(\xi_i, x, y)\right] \Delta T_{o,i}
\]
Heat flux from the surface,

\[ \dot{q}_o'' = -k \frac{\partial T}{\partial y} \bigg|_o \]

\[ \dot{q}_o'' = k \left\{ \int_o^x \left[ \theta_y^+ (\xi, x, o) \right] \frac{dT_o}{d\xi} \, d\xi + \sum_{i=1}^{k} \left[ \theta_y^+ (\xi_i, x, o) \right] \Delta T_{o,i} \right\} \]  

(7.34a)

Note for the case of a single step, \( \left\{ \frac{dT_o}{d\xi} = 0 \right\} \)

\[ \dot{q}_o'' = -k \frac{\partial T}{\partial y} \bigg|_{y=o} = k \theta_y^+ (\xi, x, o) \ast [T_o - T_\infty] \]  

(7.34b)

and since \( \dot{q}_o'' = h[T_o - T_\infty] \), eq 7.34b can be rewritten as,

\[ h (T_o - T_\infty) = k \theta_y^+ (\xi, x, o) \ast [T_o - T_\infty] \]

\[ \theta_y^+ (\xi, x, o) = \frac{h (\xi, x)}{k} \]

- So for an arbitrary wall temperature variation,

\[ \dot{q}_o'' = \int_o^x h(\xi, x) \frac{dT_o}{d\xi} \, d\xi + \sum_{i=1}^{k} h(\xi_i, x) \Delta T_{o,i} \]  

(7.34c)

where \( h(\xi,x) \) is the local convection coefficient obtained from the single step function solution.

- The previously derived solution, eq 7.31 can be rewritten as,
Substituting eq 7.35 into eq 7.34c enables the heat transfer rate \( \dot{q}_o \) for a flat plate laminar boundary layer, with an axial wall temperature distribution to be determined.

Note: The change in \( T_o \) as a function of \( \xi \), and \( \frac{dT_o}{d\xi} \) must be supplied to the integral eq 7.34c, as well as, any abrupt changes in \( T_o \) in the summation term.

**Example:**
Consider a flat plate with a step change in \( T_o \) at the leading edge, i.e., \( \xi = x = 0 \), followed by a linear variation in \( T_o \).

Find the wall heat flux?

Let the temperature profile be written as:

\[
T_o = T_\infty + a + bx
\]

and the derivative,

\[
\frac{dT_o}{dx} = b = \frac{dT_o}{d\xi}
\]

- Evaluate eq 7.34c with the step change in \( T_o \) occurring at \( \xi = 0 \).
\[
\dot{q}_o'' = \frac{0.332k}{x} \Pr^{1/3} \text{Re}^{1/2} \left\{ \int_0^x \left\{1 - (\xi/x)^{3/4}\right\}^{-1/3} b \, d\xi + a \right\}
\]

(7.36)

The above integral can be put into the form of a Beta function,

\[
\int_0^1 z^{m-1} (1 - z)^{n-1} \, dz = \beta(m, n)
\]

(7.37)

Where the Beta function is defined as;

\[
\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)}
\]

(7.38)

for \(m > 0\) and \(n < \infty\)

Simplify by letting

\[
W = 1 - \left(\frac{\xi}{x}\right)^{3/4}
\]

and

\[dW = -\frac{3}{4} \left(\frac{\xi}{x}\right)^{-1/4} \frac{d\xi}{x}\]

or

\[dW = -\frac{3}{4} \frac{1}{\left(\frac{\xi}{x}\right)^{1/4}} \frac{d\xi}{x}\]

Rearrange and solve for \(d\xi\),

\[d\xi = -\frac{4}{3} \left(\frac{\xi}{x}\right)^{1/4} x \, dW\]

Now substitute into the integral of eq 7.36 and integrate,

\[b \int_0^x W^{-1/3} \, d\xi\]
Using the original definition of $W$, and rearranging,

\[ 1 - W = \left( \frac{\xi}{x} \right)^{3/4} \]

Taking the 1/3rd power,

\[ (1 - W)^{1/3} = \left( \frac{\xi}{x} \right)^{1/4} \]

Substitute into eq 7.39

\[ \frac{4}{3} bx \int_0^1 W^{-1/3} (1 - W)^{1/3} dW \]  \hspace{1cm} (7.40)

Using the format of the Beta function, eq 7.37, set the following;

\[ m -1= -1/3 \text{ and } n -1= 1/3 \]

Evaluate the Beta function using appropriate tables for $m = 2/3$ and $n = 4/3$,

\[ \beta(m, n) = \frac{\Gamma(2/3) \Gamma(4/3)}{\Gamma(6/3)} = 1.209 \]

So eq 7.40 becomes,

\[ \dot{q}''_o = \frac{0.332k}{x} \text{Pr}^{1/3} \text{Re}^{1/2}_x \left\{ \frac{4}{3} b 1.209x + a \right\} \]

\[ \dot{q}''_o = \frac{0.332k}{x} \text{Pr}^{1/3} \text{Re}^{1/2}_x \left\{ 1.612 bx + a \right\} \]
or in terms of the local Nu,

\[ Nu_x = \frac{0.332 \Pr^{1/3} \Re_x^{1/2} \{ 1.612bx + a \}}{bx + a} \]  

(7.41)

This relation reduces to the constant wall temperature solution when \( b = 0 \).

Further if \( a = 0 \) (no step change in temperature):

\[ b \neq 0, \text{ and } \Pr = 0.7 \]  

\( \text{eq (7.41)} \) reduces to,

\[ Nu_x = 0.4752 \Re_x^{1/2} \]

This solution differs from the exact solution, obtained using a similarity transformation, by only 1.1%.

--- The End ---

7.6 Flow Along a Flat Plate with a Constant \( U_\infty \) and with an Arbitrary Specified Surface Heat Flux

In some cases the heat transfer coefficient resulting from a step function solution can be put in the following form;

\[ h(\xi, x) = f(x) \left( x^\gamma - \xi^\gamma \right)^{-\alpha} \]

Under this condition the wall temperature, as a function of heat input can be obtained.

\[ T_o(x) - T_\infty = \int_{\xi_o}^{x} \hat{q}_o''(\xi) \ g(\xi, x) \ d\xi \]  

(7.42)

Where \( \hat{q}_o''(\xi) = \hat{q}_o''(x) \) is an arbitrarily prescribed surface heat flux and \( g(\xi, x) \) is a modification of the function \( h(\xi, x) \) found from,
For a laminar boundary layer growing on a flat plate, the step function solution previously obtained is of the necessary form, and thus \( g(\xi, x) \) can be evaluated from eq 7.31,

\[
g(\xi, x) = \frac{\Pr^{-1/3} \Re^{-1/2}}{6 \Gamma(4/3) \Gamma(5/3) 0.332k} \left\{ 1 - (\xi / x)^{3/4} \right\}^{-2/3}
\]

Substituting eq 7.44 into 7.42 yields,

\[
T_o(x) - T_\infty = \frac{0.632}{k} \Pr^{-1/3} \Re^{-1/2} \int_0^x \left\{ 1 - (\xi / x)^{3/4} \right\}^{-2/3} \dot{q}_o''(\xi) \, d\xi
\]

If \( \dot{q}_o'' = \) constant; the integral in eq 7.45 takes the form of a Beta function, and the \( \text{Nu}_x \) becomes,

\[
\text{Nu}_x = 0.453 \Pr^{1/3} \Re_x^{1/2}
\]

This result is 36% higher than that determined at the same location, using eq 7.11, for a flat plate with constant \( T_\infty \). Recall that the constant \( T_\infty \) solution was obtained by a similarity analysis.

No similarity solution exists for the constant heat flux boundary condition.

Expressing this result in terms of a St, use the energy integral eq to determine the enthalpy thickness (\( \Delta_2 \))

\[
\text{St} \Pr^{4/3} = 0.205 / \text{Re} \Delta_2
\]

Note the similarity with eq 7.15 for the case of \( T_\infty = \) constant.

This suggests that when the local length scale \( \Delta_2 \) is used, the past history of the thermal boundary layer is not as important, and therefore eq 7.47 provides a good approximation.
Chapter 7 Addendum

For the case of a laminar flow over an isothermal \((T_0 = \text{constant})\) flat plate, the local Nusselt \(\#\), eq 7.11 applies,

\[
Nu_x = 0.332 \Pr^{1/3} \Re_x^{1/2} \quad (7.11)
\]

for \(\Pr\) between: \(0.6 < \Pr < 50\)

or for the average \(Nu \#\) over the plate \(L\);

\[
Nu_L = 0.664 \Pr^{1/3} \Re_L^{1/2}
\]

Good agreement is observed in the lower \(Re_L\) regime (laminar flow) when comparing eq 7.11 to experimental data.

However, eq 7.11 does not apply to fluids with very low \(Pr\), such as liquid...
metals or fluids with high Pr, such as heavy oils and silicones.

In 1973 Churchill & Ozoe correlated a large quantity of laminar flow flat plate data and developed the following relation:

\[
Nu_x = \frac{C_1}{\Pr^{1/3}} \frac{Re_x^{1/2}}{1 + \left( \frac{C_2}{Pr} \right)^{2/3} \Pr^{1/4}} ; \text{ for } Re_x \Pr > 100
\]  

(7.48)

All thermophysical properties are evaluated at the film temperature,

\[
T_f = \frac{T_\infty + T_o}{2}
\]

The following constants are to be applied to eq 7.48

<table>
<thead>
<tr>
<th>Constants</th>
<th>To = constant (isothermal)</th>
<th>Constant heat flux</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>0.3387</td>
<td>0.4637</td>
</tr>
<tr>
<td>C2</td>
<td>0.0468</td>
<td>0.0207</td>
</tr>
</tbody>
</table>

Reynolds-Colburn Analogy:

where

\[
St_x \Pr^{2/3} = \frac{C_{fx}}{2}
\]

\[
St_x = \frac{Nu_x}{Pr Re_x} = \frac{h_x}{\rho c_p U_\infty}
\]

and \(C_{fx}\) represents the local skin friction coefficient

\[
C_{fx} = \frac{\frac{1}{2} \rho U_\infty^2}{\tau_{ox}}
\]
## Summary of equations for flow over flat plates

Properties evaluated at $T_f = (T_w + T_\infty)/2$ unless otherwise noted.

<table>
<thead>
<tr>
<th>Flow regime</th>
<th>Restrictions</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laminar, local</td>
<td>$T_w = \text{const}, \operatorname{Re}_x &lt; 5 \times 10^5, \ 0.6 &lt; \Pr &lt; 50$</td>
<td>$\operatorname{Nu}_x = 0.332 \operatorname{Re}_x^{1/2} \operatorname{Pr}^{1/3}$</td>
</tr>
<tr>
<td>Laminar, local</td>
<td>$T_w = \text{const}, \operatorname{Re}_x &lt; 5 \times 10^5, \ \operatorname{Re}_x, \Pr &gt; 100$</td>
<td>$\operatorname{Nu}_x = \frac{0.3387 \operatorname{Re}_x^{1/2} \operatorname{Pr}^{1/3}}{1 + \left(\frac{0.0468}{\Pr}\right)^{2/3}}$</td>
</tr>
<tr>
<td>Laminar, local</td>
<td>$q_w = \text{const}, \operatorname{Re}_x &lt; 5 \times 10^5, \ 0.6 &lt; \Pr &lt; 50$</td>
<td>$\operatorname{Nu}_x = 0.453 \operatorname{Re}_x^{1/2} \operatorname{Pr}^{1/3}$</td>
</tr>
<tr>
<td>Laminar, local</td>
<td>$q_w = \text{const}, \operatorname{Re}_x &lt; 5 \times 10^5$</td>
<td>$\operatorname{Nu}_x = \frac{0.4637 \operatorname{Re}_x^{1/2} \operatorname{Pr}^{1/3}}{1 + \left(\frac{0.0207}{\Pr}\right)^{2/3}}$</td>
</tr>
<tr>
<td>Laminar, average</td>
<td>$\operatorname{Re}_L &lt; 5 \times 10^5, T_w = \text{const}$</td>
<td>$\overline{\operatorname{Nu}}<em>L = 2 \operatorname{Nu}</em>{x,L} = 0.664 \operatorname{Re}_L^{1/2} \operatorname{Pr}^{1/3}$</td>
</tr>
<tr>
<td>Laminar, local</td>
<td>$T_w = \text{const}, \operatorname{Re}_x &lt; 5 \times 10^5, \Pr &lt; 1$ (liquid metals)</td>
<td>$\operatorname{Nu}_x = 0.564(\operatorname{Re}_x \operatorname{Pr})^{1/3}$</td>
</tr>
<tr>
<td>Laminar, local</td>
<td>$T_w = \text{const}, \operatorname{Re}_x &lt; 5 \times 10^5, \operatorname{Pr} &lt; 50$</td>
<td>$\operatorname{Nu}_x = 0.332 \operatorname{Re}_x^{1/2} \operatorname{Pr}^{1/3} \left[1 - \left(\frac{\operatorname{Re}_x}{\operatorname{Re}_0}\right)^{2/3}\right]^{-1/3}$</td>
</tr>
<tr>
<td>Turbulent, local</td>
<td>$T_w = \text{const}, \ \operatorname{Re}_x &lt; 10^5$</td>
<td>$\operatorname{St}_x \operatorname{Pr}^{2/3} = 0.0296 \operatorname{Re}_x^{-0.2}$</td>
</tr>
<tr>
<td>Turbulent, local</td>
<td>$\operatorname{Re}_x &lt; 10^6$</td>
<td>$\operatorname{St}_x \operatorname{Pr}^{2/3} = 0.185(\log \operatorname{Re}_x)^{-2.584}$</td>
</tr>
<tr>
<td>Turbulent, local</td>
<td>$q_w = \text{const}, \ \operatorname{Re}_x &lt; 10^5$</td>
<td>$\operatorname{Nu}<em>x = 1.04 \operatorname{Nu}</em>{x,T_w=\text{const}}$</td>
</tr>
<tr>
<td>Laminar-turbulent, average</td>
<td>$\operatorname{Re}<em>x &lt; 10^7, \ \operatorname{Re}</em>{crit} = 5 \times 10^5$</td>
<td>$\overline{\operatorname{Nu}}<em>L = 0.036 \operatorname{Pr}</em>{0.43}(\operatorname{Re}<em>L^{1/2} - 9200) \left(\frac{\mu_0}{\mu</em>{\infty}}\right)^{1/4}$</td>
</tr>
<tr>
<td>Laminar-turbulent, average</td>
<td>$\operatorname{Re}<em>x &lt; 10^7, \ \operatorname{Re}</em>{crit} = 5 \times 10^5$</td>
<td>$\overline{\operatorname{Nu}}_L = 0.037 \operatorname{Re}_x^{0.2} - 871 \operatorname{Re}_L^{1/2}$</td>
</tr>
<tr>
<td>High-speed flow</td>
<td>$T_w = \text{const}, \ q = hA(T_w - T_\infty)$</td>
<td>Same as for low-speed flow with properties evaluated at $T^* = T_w + 0.5(T_w - T_\infty) + 0.22(T_{\text{aw}} - T_\infty)$</td>
</tr>
</tbody>
</table>

### Boundary layer thickness

<table>
<thead>
<tr>
<th>Flow regime</th>
<th>Restrictions</th>
<th>$\frac{\delta}{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laminar</td>
<td>$\operatorname{Re}_x &lt; 5 \times 10^5$</td>
<td>$5.0 \operatorname{Re}_x^{-1/2}$</td>
</tr>
<tr>
<td>Turbulent</td>
<td>$\operatorname{Re}_x &lt; 10^7$</td>
<td>$0.381 \operatorname{Re}_x^{-1/5}$</td>
</tr>
<tr>
<td>Mixed B.L.</td>
<td>$5 \times 10^5 &lt; \operatorname{Re}<em>x &lt; 10^7, \ \operatorname{Re}</em>{crit} = 5 \times 10^5$</td>
<td>$0.381 \operatorname{Re}_x^{-1/5} - 0.256 \operatorname{Re}_x^{-1}$</td>
</tr>
</tbody>
</table>

### Friction coefficients

<table>
<thead>
<tr>
<th>Flow regime</th>
<th>Restrictions</th>
<th>$C_{f,x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laminar, local</td>
<td>$\operatorname{Re}_x &lt; 5 \times 10^5$</td>
<td>$0.664 \operatorname{Re}_x^{-1/2}$</td>
</tr>
<tr>
<td>Turbulent, local</td>
<td>$5 \times 10^5 &lt; \operatorname{Re}_x &lt; 10^7$</td>
<td>$0.0592 \operatorname{Re}_x^{-1/5}$</td>
</tr>
<tr>
<td>Turbulent, local</td>
<td>$10^7 &lt; \operatorname{Re}_x &lt; 10^9$</td>
<td>$0.37(\log \operatorname{Re}_x)^{-2.584}$</td>
</tr>
<tr>
<td>Turbulent, average</td>
<td>$\operatorname{Re}_{crit} &lt; \operatorname{Re}_x &lt; 10^9$</td>
<td>$\overline{C}_f = 0.455 \left(\frac{(\log \operatorname{Re}_L)^{2.584} - A}{\operatorname{Re}_L}\right)$</td>
</tr>
</tbody>
</table>

$A$ from Table 5-1
Boundary Layer Flow over a Flat Plate with Viscous Dissipation:

Effects of viscous dissipation can be significant if the viscosity of the fluid is high (i.e., flow of lubricant through a journal bearing) or if the fluid velocity is high (i.e., gas flow at relatively high M#’s).

If viscous dissipation is important thermophysical property variations as f(T) must be considered. This effect can be approximated, by evaluating properties at some mean temperature.

The 2D governing eqs for a flat plate with constant $U_\infty$ and $k$ are:

$$
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \left[ \frac{\partial^2 u}{\partial y^2} \right]$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c} \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c} \left( \frac{\partial u}{\partial y} \right)^2$$

Recall, viscous dissipation is represented by the last term in the energy eq.

Velocity bcs are:

$$u = v = 0 \ @ \ y = 0$$

$$u = U_\infty \ @ \ y \geq \delta$$

Thermal bcs are:

$$q = 0 \ @ \ y = 0 \qquad \text{or} \qquad T = T_o \ @ \ y = 0$$

$$T = T_\infty \ @ \ y \geq \delta$$

The first thermal bc implies the wall is adiabatic, and from Fourier’s law @ the wall (y = 0),

$$\frac{\partial T}{\partial y} = 0$$

Since fluid properties are assumed constant, viscous dissipation has no effect on the conservation of mass and momentum. Therefore the similarity solutions found in chapter 6 are valid.
Now consider the Energy Eq.

Case 1: Adiabatic Wall

Similarity temperature profiles in the presence of viscous dissipation include the work done by the viscous forces, which results in an increased local fluid temperature.

In the adiabatic case the temperature profile is related to the fluids freestream kinetic energy, and has the following non-dimensional form:

\[
\frac{T - T_\infty}{(T_t - T_\infty)} = \theta_a^+ (\eta)
\]

\[
\frac{T - T_\infty}{(u_\infty^2 / 2 c_p)} = \theta_a^+ (\eta)
\]

Substituting this form and the corresponding velocity profile into the energy eq, one obtains, after some rearranging:

\[
\theta_a'' + \frac{\Pr}{2} F \theta_a' + 2 \Pr (F'')^2 = 0
\]

Note that both \(\theta_a^+\) depends on both \(\Pr\) and \(\eta\)

Thermal bc's are;

\[
\theta_a' = 0 \text{ @ } \eta = 0
\]

\[
\theta_a = 0 \text{ @ } \eta \geq \delta
\]

Solving the above equation:

\[
\theta_a^+(0) = \frac{\int_0^\eta \left[ \exp \left( \frac{\Pr}{2} \int_o^\eta F \, d\eta \right) 2 \Pr (F'')^2 \right] \, d\eta}{\int_0^{\infty} \left[ \exp \left( \frac{\Pr}{2} \int_o^{\infty} F \, d\eta \right) \right] \, d\eta}
\]
The above eq could be numerically integrated or one can approximate $\theta_a^+$ by, $\theta_a^+ = Pr^{1/2}$
and therefore

$$r = \frac{T_{oa} - T_\infty}{T_t - T_\infty} = \frac{T_{aw} - T_\infty}{T_t - T_\infty}$$

Where:
- $T_{aw} = T_{oa}$ is the adiabatic wall temperature
- $T_t$ is the stagnation temperature
- $T_\infty$ is the freestream static temperature
- $r$ is the recovery factor

$$\theta_a^+(0) = \frac{T_{aw} - T_\infty}{(u_\infty^2 / 2c_p)}$$

For a perfect gas:

$$\frac{T_{aw}}{T_\infty} = 1 + \theta_a^+(0) \frac{\gamma-1}{2} M_\infty^2$$

Where $\theta_a^+(0) = r$; the recovery factor, therefore,

$$r = Pr^{1/2}$$

The effect of M# on the recovery factor, for a laminar flat plate boundary layer in air is presented.
Case 2: Constant Wall Temperature ($T_0$)
Since the energy eq is linear in $T$, it was assumed that the constant $T_0$ case with viscous dissipation can be thought of as a combination of:

a) constant $T_0$ case w/o dissipation, and
b) constant heat flux case w/dissipation.

The resulting temperature profile, $\theta^+_{T,\phi}$ for the case of constant $T_0$ with viscous dissipation;

$$\theta^+_{T,\phi} = \theta^+_{aw} + [\theta^+_{T}(0) - \theta^+_{aw}(0)](1 - \theta^+_{T})$$

Where;
- $\theta^+_{aw} =$ Temperature profile for adiabatic wall w/ viscous dissipation
- $\theta^+_{T} =$ Temperature profile for $T_0 = \text{constant w/o viscous dissipation}$

and

$$\frac{q_0 x / k}{\sqrt{Re_x}} = 0.332 \text{ Pr}^{1/3} (T_o - T_o,ad)$$
\[
\frac{q_0 x}{(T_o - T_{o,ad}) k} = 0.332 \text{ Re}^{1/2} \text{ Pr}^{1/3}
\]

Defining the local \( \text{Nu}_x \);

\[
\frac{q_0 x}{(T_o - T_{o,ad}) k} = \text{Nu}_x
\]

one can write, \( \text{Nu}_x = 0.332 \text{ Re}^{1/2} \text{ Pr}^{1/3} \)

or integrating over the plate length the average \( \text{Nu}_L \) can be determined,

\[
\text{Nu}_L = 0.332 \text{ Re}^{1/2} \text{ Pr}^{1/3}
\]
Example:
Air flows over a flat plate @ constant \( T_o \) (= 30 °C). Determine if heat is being transferred to or from the plate.

Assume a laminar boundary layer with \( \text{Pr} = 0.7 \) and \( \gamma = 1.4 \)

Recall the recovery factor; \( r = \text{Pr}^{1/2} = 0.837 \)

Solution:

\[
\frac{T_{aw}}{T_\infty} = 1 + r \frac{\gamma - 1}{2} M_\infty^2
\]

\[
\frac{T_{aw}}{T_\infty} = 1 + r \frac{0.2}{2} M_\infty^2
\]

\[
\frac{T_{aw}}{T_\infty} = 1.135
\]

\( T_{aw} = 309.9K = 36.9°C \)

\( T_{aw} = 309.9K = 36.9°C \)

Therefore, heat is being transferred from the air to the plate!!!!

--- The End ---
Effect of Fluid Property Variations w/ Viscous Dissipation

Since large temperature variations can occur when viscous dissipation is present, changes in thermophysical properties must be considered.

When viscous dissipation is not of concern, a mean film temperature can be used to evaluate properties: \( (T_w + T_\infty)/2 \)

However, when viscous dissipation is important there are 3 temperatures \( (T_\infty, T_w, \text{ and } T_{aw}) \) that affect fluid properties and hence, the temperature distribution.

The adjacent figure displays Temperature distributions in a Laminar boundary layer with and w/o viscous dissipation.

Fluid property variations can be accounted for using the following relation,

\[
T_{\text{prop}} = T_\infty + 0.5(T_0 - T_\infty) + 0.22(T_{aw} - T_\infty)
\]