Chapter 2C: Eulers’ Equation of Motion

2. Eulers’ Equation of Motion

• These equations will be developed in 2D and only consider body and pressure forces (i.e., neglecting viscosity).

• Newton’s Law (x-component)

\[ F_x = m \ a_x \]  

(2.7)

where

- \( F_x \) = forces acting on the fluid particle in the x-direction
- \( m \) = mass of fluid particle
- \( a_x \) = acceleration of the fluid particle in the x-direction

• This equation has been written for an inertial reference frame, therefore the reference frame should not be rotated or accelerated, it must move at constant velocity only.

Note: Non-inertial reference frames can be handled by the addition of the appropriate terms including the Coriolis and centrifugal force.
Chapter 2C: Eulers’ Equation of Motion

- Let’s consider a fluid particle at two different time steps i.e., a Lagrangian approach;

\[
\begin{align*}
\text{where } u &= f(x, y, t) \text{ and } a_x = \frac{du}{dt} \\
\text{Using rules of partial differentiation,}\end{align*}
\]

\[
\begin{align*}
\frac{du}{dt} &= \frac{\partial u}{\partial t} + \frac{u}{\partial x} \frac{dx}{dt} + \frac{v}{\partial y} \frac{dy}{dt} \\
\frac{dv}{dt} &= \frac{\partial v}{\partial t} + \frac{u}{\partial x} \frac{dx}{dt} + \frac{v}{\partial y} \frac{dy}{dt} \\
\frac{dw}{dt} &= \frac{\partial w}{\partial t} + \frac{u}{\partial x} \frac{dx}{dt} + \frac{v}{\partial y} \frac{dy}{dt}
\end{align*}
\]

and,

\[
\begin{align*}
\frac{du}{dt} &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \\
\frac{dv}{dt} &= \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \\
\frac{dw}{dt} &= \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y}
\end{align*}
\]

Recalling the definition of a total or substantial derivative,

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}
\]

- The rate of change of any vector or scalar property (q), in 3D can be written as,

\[
\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \frac{u}{\partial x} \frac{\partial q}{\partial x} + \frac{v}{\partial y} \frac{\partial q}{\partial y} + \frac{w}{\partial z} \frac{\partial q}{\partial z}
\]

\[
\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \nabla \cdot \mathbf{q}
\]

- If we let q equal the local velocity, this results in

\[
\begin{align*}
a_x &= \frac{Du}{Dt} \quad \text{and} \quad a_y = \frac{Dv}{Dt} \quad \text{and} \quad a_z = \frac{Dw}{Dt}
\end{align*}
\]

or in vector form,

\[
\mathbf{a} = \frac{D}{Dt} \nabla \quad \text{(2.9)}
\]
Chapter 2C: Eulers’ Equation of Motion

- The acceleration in Eq. (2.9) is a result of the forces acting on the fluid particle.

\[ a_x = \frac{Du}{Dt} \quad \text{and} \quad F_x = m \frac{Du}{Dt} = -\frac{D(mu)}{Dt} \quad \text{For a constant mass} \]

\[ \sum \text{forces acting on a fluid particle} = \text{time rate of change of linear momentum} \]

- Forces to be considered are:
  - a) pressure acting on the surfaces
  - b) forces acting directly on the fluid particle (mass).

Note: These are body forces \( f_x \), whose magnitude is mass dependent (force/unit mass).

- Now examine the forces on a 2D fluid element (Fig. 2.6).

\[
\sum \text{forces acting on a fluid particle} = \frac{\partial }{\partial x} \left( \rho \frac{Du}{Dt} \right) + \frac{\partial }{\partial y} \left( \rho \frac{Dv}{Dt} \right) + \frac{\partial }{\partial z} \left( \rho \frac{Dw}{Dt} \right)
\]

\[
\frac{\partial }{\partial x} \left( \rho \frac{Du}{Dt} \right) = F_x = -\frac{\partial }{\partial x} \left( \rho \frac{Du}{Dt} \right) + \rho f_x dx dy dz
\]

where term “a” represents the surface force & term “b” the body force.

- In the x-direction: \[ \frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + f_x \] (2.10a)

- In the y-direction: \[ \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + f_y \] (2.10b)
Chapter 2C: Eulers’ Equation of Motion

• Including the z-direction term, \( w \frac{\partial}{\partial z} \) in the material derivative,

\[
\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + f_z \tag{2.10c}
\]

• Therefore, Eqs. 2.10a-c describe the motion of a perfect or ideal (i.e., inviscid) fluid in an inertial reference frame.

• These equations are known as the Euler equations and apply to both compressible and incompressible fluids.

• Now lets re-derive Euler’s equations in vector form.

• Consider a mass within some volume (\( V \)), using Newton’s Law the acceleration term becomes,

\[
\vec{F} = \int_V \rho \frac{D\vec{V}}{Dt} \, dV
\]

where body and pressure forces in three dimensions are given as

\[
\int_\mathcal{V} \rho \vec{f} \, d\mathcal{V} - \int_\mathcal{S} p(\hat{n} \, dS)
\]

and \( \hat{n} \) is the unit vector normal to the surface element \( dS \).

• Using Gauss’s Theorem, transform the surface integral into a volume integral,

\[
\int_\mathcal{V} \nabla \phi \, d\mathcal{V} = \int_\mathcal{V} \phi \, (\nabla \cdot \vec{dS})
\]

Collecting terms,

\[
\int_\mathcal{V} \rho \frac{D\vec{V}}{Dt} \, d\mathcal{V} = \int_\mathcal{V} \rho \vec{f} \, d\mathcal{V} - \int_\mathcal{V} \nabla p \, d\mathcal{V}
\]

\[
\int_\mathcal{V} \left( \rho \frac{D\vec{V}}{Dt} + \nabla p - \rho \vec{f} \right) \, d\mathcal{V} = 0
\]
Chapter 2C: Eulers’ Equation of Motion

- Since this integral must hold for all arbitrary volume elements

\[
\left( \rho \frac{DV}{Dt} + \nabla p - \rho \vec{f} \right) = 0
\]

\[
\frac{DV}{Dt} = - \frac{1}{\rho} \nabla p + \vec{f}
\]  \hspace{1cm} (2.11)

expanding the material derivative

\[
\frac{DV}{Dt} = \frac{\partial V}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} = - \frac{1}{\rho} \nabla p + \vec{f}
\]  \hspace{1cm} (2.12)

**Term:**
1. Total rate of change of momentum
2. Temporal rate of change of momentum in the CV
3. Convective rate of change of momentum (convective acceleration)
4. Surface force
5. Body force