Chapter 7b, Torsion

\[ \tau = 0 \]

straight lines in the cross section (cross sectional projection) remain straight

the whole cross section (cross sectional projection) rotates by the same amount \( \theta(x, y, z) = \theta(x) = \beta x \)
Warping Function

Saint-Venant theory

\[ \Psi = \Psi(y,z) \text{ is the so-called warping function} \]

\[ u = \beta \Psi(y,z), \quad v = -\theta z = -\beta x z, \quad w = \theta y = \beta x y \]

\[ \beta = \frac{d\theta}{dx} \]

\[ \frac{\partial \varepsilon_x}{\partial x} = 0, \quad \varepsilon_y = \frac{\partial v}{\partial y} = 0, \quad \varepsilon_z = \frac{\partial w}{\partial z} = 0, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 \]

\[ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \beta(\frac{\partial \Psi}{\partial y} - z) \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \beta(\frac{\partial \Psi}{\partial z} + y) \]

\[ \tau_{xy} = G\beta(\frac{\partial \Psi}{\partial y} - z), \quad \tau_{xz} = G\beta(\frac{\partial \Psi}{\partial z} + y) \]

\[ \overline{OA} = \sqrt{y^2 + z^2} = r, \quad \frac{y}{r} = \frac{w}{r \theta}, \quad \frac{z}{r} = \frac{-v}{r \theta} \]
compatibility relationship

automatic (started from continuous, differentiable displacements)

equilibrium equations

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \\
0 + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \\
0 + 0 + 0 = 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \\
0 + 0 + 0 = 0 \\
\frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = G \beta \left[ \frac{\partial}{\partial y} \left( \frac{\partial \Psi}{\partial y} - z \right) + \frac{\partial}{\partial z} \left( \frac{\partial \Psi}{\partial z} + y \right) \right] \\
\frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = 0 \quad \nabla^2 \Psi = 0
\]

boundary conditions

boundary normal unit vector
\[ \mathbf{n} = n_y \mathbf{j} + n_z \mathbf{k} = \frac{dz}{ds} \mathbf{j} - \frac{dy}{ds} \mathbf{k} \]

boundary tangent unit vector
\[ \mathbf{s} = s_y \mathbf{j} + s_z \mathbf{k} = \frac{dy}{ds} \mathbf{j} + \frac{dz}{ds} \mathbf{k} \]
shear traction \[ dT = dT_y \mathbf{j} + dT_z \mathbf{k} = dA(\tau_{xy} \mathbf{j} + \tau_{xz} \mathbf{k}) \]
\[ dT \cdot \mathbf{n} = 0, \quad \text{i.e.,} \quad (\tau_{xy} \mathbf{j} + \tau_{xz} \mathbf{k}) \cdot \left( \frac{dz}{ds} \mathbf{j} - \frac{dy}{ds} \mathbf{k} \right) = \tau_{xy} \frac{dz}{ds} - \tau_{xz} \frac{dy}{ds} = 0 \]
\[ dT \times \mathbf{s} = 0, \quad \text{i.e.,} \quad (\tau_{xy} \mathbf{j} + \tau_{xz} \mathbf{k}) \times \left( \frac{dy}{ds} \mathbf{j} + \frac{dz}{ds} \mathbf{k} \right) = i(\tau_{xy} \frac{dz}{ds} - \tau_{xz} \frac{dy}{ds}) = 0 \]
\[ \frac{\tau_{xy}}{dz} = \frac{\tau_{xz}}{dy} = 0 \]
\[ \tau_{xy} = G\beta \left( \frac{\partial \Psi}{\partial y} - z \right) \quad \tau_{xz} = G\beta \left( \frac{\partial \Psi}{\partial z} + y \right) \]
\[ \frac{\partial \Psi}{\partial y} \frac{dz}{ds} - \frac{\partial \Psi}{\partial z} \frac{dy}{ds} = y \frac{dy}{ds} + z \frac{dz}{ds} \]
\[ \frac{\partial \Psi}{\partial y} \frac{dz}{ds} - \frac{\partial \Psi}{\partial z} \frac{dy}{ds} = \frac{1}{2} \frac{d}{ds} \left( y^2 + z^2 \right) \]
\[ \text{torque} \]
\[ T = \iint_A (\tau_{xz} y - \tau_{xy} z) \, dA \]
\[ T = G\beta \iint_A \left( \frac{\partial \Psi}{\partial z} y + y^2 - \frac{\partial \Psi}{\partial y} z + z^2 \right) \, dA \]
\[ T = G\beta \iint_A \left( \frac{\partial \Psi}{\partial z} y - \frac{\partial \Psi}{\partial y} z \right) \, dA + G\beta J \]
\[ J = \iint_A (y^2 + z^2) \, dA = I_z + I_y \]
Summary

Saint-Venant warping function, $\Psi$

shear stress  
$$\tau_{xy} = G\beta \left( \frac{\partial \Psi}{\partial y} - z \right), \quad \tau_{xz} = G\beta \left( \frac{\partial \Psi}{\partial z} + y \right)$$

compatibility relationship  automatic

equilibrium equations  \n$$\nabla^2 \Psi = 0
$$
boundary conditions  
$$\tau_{xy} \frac{dz}{ds} - \tau_{xz} \frac{dy}{ds} = 0$$  
$$\frac{\partial \Psi}{\partial y} \frac{dz}{ds} - \frac{\partial \Psi}{\partial z} \frac{dy}{ds} = \frac{1}{2} \frac{d}{ds} (y^2 + z^2)$$

torque  
$$T = \oint (\tau_{xz} y - \tau_{xy} z) dA$$

Prandtl stress function, $\phi$

shear stress  
$$\tau_{xy} = \frac{\partial \phi}{\partial z}, \quad \tau_{xz} = -\frac{\partial \phi}{\partial y}$$

compatibility relationship  \n$$\nabla^2 \phi = -2G\beta$$

equilibrium equations  automatic
boundary conditions  
$$d\phi = 0$$
torque  
$$T = 2\oint \phi dA$$
Saint-Venant Warping Function for an Elliptical Cross Section

equilibrium equation
\[ \nabla^2 \Psi = 0 \]

warping function
\[ \Psi = Ayz \]

shear stresses
\[ \tau_{xy} = G\beta \left( \frac{\partial \Psi}{\partial y} - z \right) = G\beta (A - 1)z \]
\[ \tau_{xz} = G\beta \left( \frac{\partial \Psi}{\partial z} + y \right) = G\beta (A + 1)y \]

boundary condition
\[ \frac{\tau_{xy}}{\tau_{xz}} = \frac{dy}{dz} \]
\[ \frac{\tau_{xy}}{\tau_{xz}} = \frac{A - 1}{A + 1} \frac{z}{y} \]

\[ \frac{y^2}{a^2} + \frac{z^2}{b^2} = f(y, z) = 1 \Rightarrow df = \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \frac{2y}{a^2} dy + \frac{2z}{b^2} dz = 0 \]
\[ \frac{dy}{dz} = -\frac{a^2}{b^2} \frac{z}{y} \]
\[
\frac{a^2}{b^2} = \frac{A - 1}{A + 1}
\]

\[
A = -\frac{a^2 - b^2}{a^2 + b^2}
\]

\[
\Psi = -\frac{a^2 - b^2}{a^2 + b^2} yz
\]

shear stress components

\[
\tau_{xy} = G\beta \left( \frac{\partial \Psi}{\partial y} - z \right) = -\frac{2G\beta a^2 z}{a^2 + b^2}
\]

\[
\tau_{xz} = G\beta \left( \frac{\partial \Psi}{\partial z} + y \right) = \frac{2G\beta b^2 y}{a^2 + b^2}
\]

shear stress

\[
\tau^2 = \tau_{xy}^2 + \tau_{xz}^2 = \left( \frac{2G\beta a^2 z}{a^2 + b^2} \right)^2 + \left( \frac{2G\beta b^2 y}{a^2 + b^2} \right)^2 = \left( \frac{2G\beta}{a^2 + b^2} \right)^2 \left[ \left( a^2 z \right)^2 + \left( b^2 y \right)^2 \right]
\]

\[
\tau^2 = \tau_0^2 \left( \frac{y^2}{a^2} + \frac{z^2}{b^2} \right)
\]

\[
\tau_0 = \frac{2G\beta a^{3/2} b^{3/2}}{a^2 + b^2}, \quad a' = \frac{a^3}{b}, \quad b' = \frac{b^3}{a}
\]

maximum shear stress

\[
\tau_{\text{max}} = \max \{ \tau_{xy} \} = \frac{2G\beta a^2 b}{a^2 + b^2}
\]
torque

\[ T = \int \int_A (\tau_{xz} y - \tau_{xy} z)\,dA \]

\[ T = \int \int_A \left( \frac{2G\beta b^2}{a^2 + b^2} y^2 + \frac{2G\beta a^2}{a^2 + b^2} z^2 \right)\,dA = \frac{2G\beta b^2}{a^2 + b^2} I_z + \frac{2G\beta a^2}{a^2 + b^2} I_y \]

\[ I_y = \frac{\pi b^4}{4} a, \quad I_z = \frac{\pi a^4}{4} b \]

\[ T = G\beta \frac{\pi a^3 b^3}{a^2 + b^2} = G\beta J_t \]

\[ J_t = \frac{\pi a^3 b^3}{a^2 + b^2} \]

\[ \frac{G\beta}{a^2 + b^2} = \frac{T}{\pi a^3 b^3} \]

\[ \tau_{xy} = -\frac{2Tz}{\pi ab^3}, \quad \tau_{xz} = \frac{2Ty}{\pi a^3 b}, \quad \tau_0 = \frac{2T}{\pi a^{3/2} b^{3/2}}, \quad \text{and} \quad \tau_{\text{max}} = \frac{2T}{\pi ab^2} \]

circular cross section \((a = b)\)

\[ J_t = \frac{\pi a^4}{2} = J \]

\[ \tau = \tau_0 \sqrt{\frac{y^2}{a^2} + \frac{z^2}{b^2}} = \frac{2T}{\pi a^4} \sqrt{\frac{y^2 + z^2}{a^2}} \]

\[ \tau = \frac{T r}{J}, \quad \text{where} \quad r = \sqrt{y^2 + z^2} \]
Prandtl Stress Function

\[ \phi = \phi(y, z) \]

shear stress

\[ \tau_{xy} = \frac{\partial \phi}{\partial z}, \quad \tau_{xz} = -\frac{\partial \phi}{\partial y} \]

equilibrium equations

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= 0 \\
0 + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_y}{\partial z} &= 0 \\
0 + 0 + 0 &= 0 \\
\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} = 0
\end{align*}
\]

automatic!

compatibility relationship

three displacements!

indirect solution using the Saint-Venant warping function

\[ \tau_{xy} = G\beta \left( \frac{\partial \Psi}{\partial y} - z \right) \quad \text{and} \quad \tau_{xz} = G\beta \left( \frac{\partial \Psi}{\partial z} + y \right) \]

\[
\begin{align*}
\frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_{yz}}{\partial z} &= G\beta \left( \frac{\partial^2 \Psi}{\partial z \partial y} - 1 - \frac{\partial^2 \Psi}{\partial y \partial z} - 1 \right) = -2G\beta \\
\tau_{xy} &= \frac{\partial \phi}{\partial z}, \quad \tau_{xz} = -\frac{\partial \phi}{\partial y} \\
\nabla^2 \phi &= -2G\beta
\end{align*}
\]
boundary conditions

\[
\tau_{xy} \frac{dz}{ds} - \tau_{xz} \frac{dy}{ds} = 0
\]

\[
\frac{\partial \phi}{\partial z} dz + \frac{\partial \phi}{\partial y} dy = d\phi = 0
\]

\[
\phi = \text{constant}
\]

for a solid cross section with closed loop boundary line one can choose

\[
\phi = 0
\]

torque

\[
T = \iint_A (\tau_{xz} y - \tau_{xy} z) dA = -\iint_A \frac{\partial \phi}{\partial y} y dA - \iint_A \frac{\partial \phi}{\partial z} z dA
\]

For a solid cross section with closed loop boundary line and \( \phi = 0 \) on the boundary:

\[
\iint_A \frac{\partial \phi}{\partial y} y dA = \int_C^D \left( \int_C^B \frac{\partial \phi}{\partial y} y dy \right) dz = \int_C^D \left\{ [\phi y]_B^B - [\phi dy]_A^A \right\} dz = -\int_C^D \int_A^B \phi dy dz = -\iint_A \phi dA
\]

\[
\iint_A \frac{\partial \phi}{\partial z} z dA = \int_A^B \left( \int_A^D \frac{\partial \phi}{\partial z} z dz \right) dy = \int_A^B \left\{ [\phi z]_C^D - [\phi dz]_A^A \right\} dy = -\int_A^B \int_C^D \phi dz dy = -\iint_A \phi dA
\]

\[
T = 2 \iint_A \phi dA
\]
If \( \phi = \phi_0 \neq 0 \) on \( s \):

\[
T = 2 \iint_A (\phi - \phi_0) \, dA
\]

When the cross section is bordered by multiple discontinuous boundary lines, \( T = 2 \iint_A \phi \, dA \) cannot be used directly.

Assume that the actual cross section \( A = A_1 - A_2 \) is divided into solid cross sections \( A_1 \) and \( A_2 \) bordered by \( s_1 \) and \( s_2 \), respectively, and both \( s_1 \) and \( s_2 \) are constant contour lines of the same stress function \( \phi \):

\[
\begin{align*}
\phi &= 0 \text{ on } s_1 \text{ and } \phi = \phi_0 \text{ on } s_2 \\
T_1 &= 2 \iint_{A_1} \phi \, dA, \quad T_2 = 2 \iint_{A_2} (\phi - \phi_0) \, dA \\
T &= T_1 - T_2 = 2 \iint_{A_1} \phi \, dA - 2 \iint_{A_2} (\phi - \phi_0) \, dA \\
&= 2 \iint_A \phi \, dA + 2 \phi_0 A_2
\end{align*}
\]
**Prandtl Stress Function for an Elliptical Cross Section**

**boundary equation**

\[
\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1
\]

**boundary condition**

\[
\phi = 0
\]

\[
\phi = C \left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right)
\]

**compatibility relationship**

\[
\nabla^2 \phi = -2C \left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -2G \beta \quad \Rightarrow \quad C = \frac{G \beta a^2 b^2}{a^2 + b^2}
\]

\[
\phi = \frac{G \beta a^2 b^2}{a^2 + b^2} \left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right)
\]

\[
\tau_{xy} = -\frac{\partial \phi}{\partial z} = -\frac{2G \beta a^2 z}{a^2 + b^2} \quad \text{and} \quad \tau_{xz} = -\frac{\partial \phi}{\partial y} = \frac{2G \beta b^2 y}{a^2 + b^2}
\]

**torque**

\[
T = 2\iint_A \phi \, dA = \frac{2G \beta a^2 b^2}{a^2 + b^2} \iint_A \left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right) \, dA
\]

\[
T = \frac{2G \beta a^2 b^2}{a^2 + b^2} \left(A - \frac{1}{a^2} I_x - \frac{1}{b^2} I_y\right) = \frac{2G \beta a^2 b^2}{a^2 + b^2} \left(\pi a b - \frac{\pi a^4}{4} a - \frac{\pi b^4}{4} a\right)
\]

\[
T = \frac{\pi G \beta a^3 b^3}{a^2 + b^2} \quad \Rightarrow \quad \frac{G \beta}{a^2 + b^2} = \frac{T}{\pi a^3 b^3}
\]

\[
\tau_{xy} = -\frac{2T z}{\pi a b^3} \quad \text{and} \quad \tau_{xz} = \frac{2T y}{\pi a^3 b}
\]
Prandtl Stress Function for an Equilateral Triangle Cross Section

boundary equations

\[ y = -\frac{\sqrt{3}}{6}a \]

\[ z = \frac{\sqrt{3}}{3}y - \frac{1}{3}a \]

\[ z = -\frac{\sqrt{3}}{3}y + \frac{1}{3}a \]

boundary condition

\[ \phi = 0 \]

\[ \phi = k(2\sqrt{3}y + a)(3z - \sqrt{3}y + a)(3z + \sqrt{3}y - a) \]

where \( k \) is an unknown constant to be determined from \( \nabla^2\phi = -2G\beta \).

\[ \phi = k(2\sqrt{3}y + a)[9z^2 - (\sqrt{3}y - a)^2] \]

\[ \phi = k(18\sqrt{3}yz^2 - 6\sqrt{3}y^3 + 9ay^2 + 9az^2 - a^3) \]

\[ \nabla^2\phi = k(-36\sqrt{3}y + 18a + 36\sqrt{3}y + 18a) \]

\[ \nabla^2\phi = 36ak = -2G\beta \]

\[ k = -\frac{G\beta}{18a} \]
torque

\[ T = 2 \iiint_a^{a/\sqrt{3}} \int \int_{y=-a/2\sqrt{3}}^{a/3-y/\sqrt{3}} \int_{z=0}^{a/3} \phi \, dz \, dy \]

\[ T = \frac{\sqrt{3}}{80} a^4 G \beta \]

\[ k = -\frac{G \beta}{18a} = -\frac{40T}{9\sqrt{3}a^5} \]

\[ \phi = -\frac{40T}{9\sqrt{3}a^5} (2\sqrt{3}y + a)(9z^2 - (\sqrt{3}y - a)^2) \]

stress

\[ \tau_{xz} = \frac{\partial \phi}{\partial y} = \frac{40T}{9\sqrt{3}a^5} \left[ 2\sqrt{3}[9z^2 - (\sqrt{3}y - a)^2] - 2\sqrt{3}(2\sqrt{3}y + a)(\sqrt{3}y - a) \right] \]

\[ \tau_{xz} = \frac{80T}{9a^5}[9z^2 - (\sqrt{3}y - a)^2 - (2\sqrt{3}y + a)(\sqrt{3}y - a)] \]

\[ \tau_{xz} = \frac{80T}{a^5}(z^2 - y^2 + \frac{ya}{\sqrt{3}}) \]

\[ \tau_{xy} = \frac{\partial \phi}{\partial z} = -\frac{40T}{9\sqrt{3}a^5} (2\sqrt{3}y + a) \frac{\partial \phi}{\partial z} [9z^2 - (\sqrt{3}y - a)^2] \]

\[ \tau_{xy} = -\frac{80T}{\sqrt{3}a^5}(2\sqrt{3}y + a)z \]
Torsional Stiffness

\[ k = \frac{T}{\beta} = GJ_t \]

prismatic bar of elliptical cross section:

\[ k = \frac{\pi Ga^3b^3}{a^2 + b^2} \]

Saint-Venant's approximate formula for prismatic bars of arbitrary cross section

\[ k \approx k_{SV} = \frac{GA^4}{4\pi^2J} \]

prismatic bar of elliptical cross section:

\[ A = \pi ab \]

\[ J = I_y + I_z = \frac{1}{4}\pi ab(a^2 + b^2) \]

\[ k_{SV} = \frac{G\pi^4a^4b^4}{4\pi^2\frac{1}{4}\pi ab(a^2 + b^2)} = \frac{\pi Ga^3b^3}{a^2 + b^2} = k \]

prismatic bar of equilateral triangle cross section:

\[ A = \frac{\sqrt{3}}{4}a^2 \]

\[ J = \frac{\sqrt{3}5a^4}{192} \]

\[ k_{SV} = \frac{GA^4}{4\pi^2J} = \frac{9\sqrt{3}Ga^4}{80\pi^2} \approx 0.0197G a^4 \]
\[ k = \frac{T}{\beta} = \frac{\sqrt{3}}{80} Ga^4 \approx 0.0217 Ga^4 \]

**prismatic bar of square cross section:**

\[ A = a^2 \]

\[ J = I_y + I_z = \frac{a^4}{6} \]

\[ k_{SV} = \frac{3Ga^4}{2\pi^2} = 0.152 Ga^4 \]

\[ k = 0.141 Ga^4 \]

**thin prismatic bar of rectangular cross section \((a >> b)\):**

\[ A = ab \]

\[ J = I_y + I_z \approx \frac{a^3b}{12} \]

\[ k_{SV} = \frac{3Gab^3}{\pi^2} = 0.304 Gab^3 \]

\[ k = 0.333 Gab^3 \]