LETTERS TO THE EDITOR

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Simplified expressions for the displacements and stresses produced by the Rayleigh wave

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(Received 28 May 1998; accepted for publication 20 August 1998)

In many cases involving Rayleigh wave propagation along the free surface of a solid, the ratio between the normal and transverse displacement components, i.e., the aspect ratio of the elliptical particle trajectory on the surface, is of great importance. Surprisingly, this aspect ratio is exactly the same as the ratio between the vector and scalar displacement potential amplitudes. More importantly, the aspect ratio is also identically the same as the ratio between the shear and normal stresses in any plane parallel to the surface at any depth, i.e., exactly the same ratio also shows up between two fundamental physical quantities. It is shown that the identical amplitude ratio, which apparently has not been pointed out in the otherwise very comprehensive literature on Rayleigh wave propagation, is a direct result of the requirement that just below the surface even the oscillating power, i.e., the imaginary component of the Poynting vector, be parallel to the surface (of course, the average power, i.e., the real component of the Poynting vector is everywhere parallel to the surface). [S0001-4966(98)06211-0]

PACS numbers: 43.20.Jr, 43.20.Mv, 43.20.Bi [DEC]

INTRODUCTION

The classical Rayleigh wave propagating on the free surface of an infinite isotropic, perfectly elastic half-space is undoubtedly one of the best known canonical problems in acoustics of solids and it is a familiar example discussed in great detail in every textbook on the subject. It is very well known that such a wave causes the particles on the surface of the infinite half-space to undergo a counterclockwise, elliptically polarized motion, as shown in Fig. 1, with the normal displacement component being larger that the transverse one.[1–10] The ratio between the two displacement components, i.e., the aspect ratio of the elliptical trajectory of the particle on the surface, is of great practical importance in many situations. For example, when the solid is immersed in fluid, the phase velocity slightly increases and the surface wave becomes strongly attenuated by leaking into the fluid. The sole source of acoustic loading by an inviscid fluid is the normal component of the surface vibration. It was recently shown that the aspect ratio of the elliptical particle trajectory produced by the dispersive Rayleigh-type surface wave on a curved surface is lower than that of the nondispersive wave on a flat surface, which results in a perceivably lower leaky attenuation.[11] Nagy and Kent used the leakage-induced attenuation, which is determined by the ratio between the normal and transverse vibration amplitudes at the surface of a rod, to assess its Poisson's ratio.[12] In the case of acoustic loading by a viscous fluid the tangential surface vibration causes additional coupling between the solid and the fluid. The aspect ratio of the surface displacement plays an important role in the acoustic loading of both rods and plates immersed in a viscous fluid and can be used to explain the sharp minimum observed in the viscosity-induced attenuation curves for such configurations.[13,14] Another area where the ratio between the tangential and normal surface displacements plays a significant role is the generation and detection of Rayleigh waves. Some excitation techniques, e.g., the so-called ''edge'' technique,[3] take advantage of the stronger normal displacement component to excite the Rayleigh wave. Laser interferometric detection is, in general, much more sensitive to the normal displacement of the surface than to the tangential component.[15–17] This implies that the displacement ratio has a crucial role in the detection of Rayleigh waves using laser based ultrasonic techniques.

In the following we briefly review the well-known analytical results for the relative amplitude of the displacements and stresses produced by the Rayleigh wave propagating on

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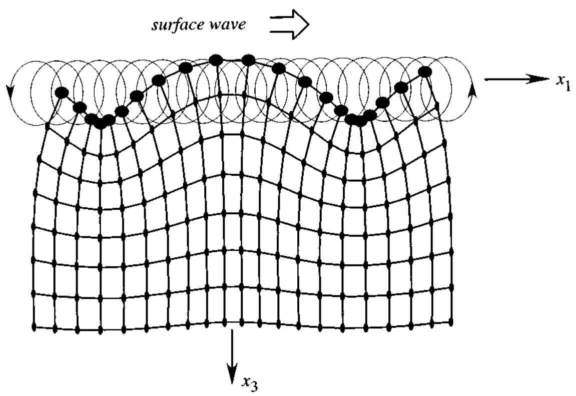

FIG. 1. Schematic diagram illustrating the particle's motion in a Rayleigh wave propagating on the free surface of an infinite half-space.

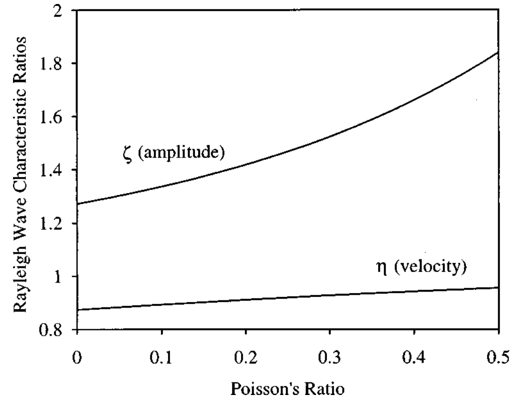

FIG. 2. Rayleigh wave characteristic ratios versus Poisson's ratio.

the free surface of an infinite half-space. By comparing these ratios, we observed an unexpected coincidence which, to the best of our knowledge, has not been pointed out in the literature yet and prompted us to write this Letter. These ratios are routinely presented in textbooks on this subject as unrelated parameters so that their identical nature remains hidden by their different algebraic forms.[1–10] We felt that the uniform presentation of these ratios could better elucidate the relationship between potentials, displacements, and stresses and, in a very modest way, might contribute to the better understanding of Rayleigh wave propagation.

I. REVIEW OF THE ANALYTICAL RESULTS

Let us consider a harmonic Rayleigh wave propagating on the free surface of an isotropic, elastic half-space. The schematic diagram illustrating the deformation in the near-surface region of the solid as well as the elliptical particle trajectory at the surface was shown in Fig. 1. The solid half-space occupies the region $x_3 \geqslant 0$ and the Rayleigh wave propagates along the x_1 axis. A common technique is to write the particle displacement vector **u** in terms of a scalar ϕ and a vector $\boldsymbol{\psi}$ potentials as follows:

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\psi}. \tag{1}$$

In an infinite isotropic solid, this representation of the displacement vector separates the wave equation into uncoupled shear and dilatational components. In contrast, at the free surface of a solid half-space the scalar and vector potentials are coupled by the requirement that together they satisfy the stress-free boundary conditions:

$$\phi = A e^{-\kappa_d x_3} e^{i(k_R x_1 - \omega t)} \tag{2}$$

and

$$\psi = -i \zeta A e^{-\kappa_s x_3} e^{i(k_R x_1 - \omega t)}, \tag{3}$$

where A is the arbitrary potential amplitude, t denotes time, ω denotes the angular frequency, k_R is the wave number of the Rayleigh wave, and $\kappa_s = \sqrt{k_R^2 - k_s^2}$ and $\kappa_d = \sqrt{k_R^2 - k_d^2}$ are decay factors related to the shear k_s and longitudinal k_d wave numbers, respectively. Finally, $\zeta = \sqrt{\kappa_d / \kappa_s}$ denotes the magnitude of the amplitude ratio between the vector and scalar potentials, which is a negative pure imaginary number. Like the characteristic velocity ratio $\eta = c_R / c_s = k_s / k_R$ between the Rayleigh and shear waves, which can be calculated by solving the well-known dispersion equation

$$(\kappa_s^2 + k_R^2)^2 - 4 \kappa_s \kappa_d k_R^2 = 0, \tag{4}$$

and is always cited in textbooks, the amplitude ratio ζ is also a function of Poisson's ratio only. Figure 2 shows the familiar curve for the normalized Rayleigh wave velocity, which ranges between 0.87 and 0.95, along with the less familiar amplitude ratio, which ranges between 1.3 and 1.8. For typical solids of Poisson's ratio around 0.3, the amplitude ratio is approximately 1.5.

Of course all this is well known and hardly new to anybody interested in acoustics. However, the authors were sincerely surprised to find out upon closer inspection that the amplitude ratio ζ is identically the same as (i) the ratio between the normal u_3 and tangential u_1 particle displacements, i.e., the aspect ratio of the elliptical particle trajectory, on the surface as well as (ii) the ratio between the shear τ_{13} and normal τ_{33} stresses at any depth.

Without the common $e^{i(k_R x_1 - \omega t)}$ term, the normal and tangential displacement components can be written as follows:

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3} = i A (k_R e^{-\kappa_d x_3} - \zeta \kappa_s e^{-\kappa_s x_3}) \tag{5}$$

and

$$u_3 = \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1} = -A (\kappa_d e^{-\kappa_d x_3} - \zeta k_R e^{-\kappa_s x_3}). \tag{6}$$

By specializing Eqs. (5) and (6) to the surface ($x_3 = 0$) and utilizing the dispersion equation given in Eq. (4) to simplify the result, we obtain

$$\left. \frac{u_3}{u_1} \right|_{x_3=0} = i \frac{\kappa_d - \zeta k_R}{k_R - \zeta \kappa_s} = -i \zeta. \tag{7}$$

Unexpectedly, the ratio of the normal and transverse displacement components on the surface turns out to be the same as the amplitude ratio between the vector and scalar potentials. Even more surprisingly, the same ratio also shows up between the shear τ_{13} and normal τ_3 stresses that can be written as follows:

$$\tau_{13} = -i A \mu 2 k_R \kappa_d (e^{-\kappa_d x_3} - e^{-\kappa_s x_3}) \tag{8}$$

and

$$\tau_{33}=A\mu(k_R^2+\kappa_s^2)(e^{-\kappa_d x_3}-e^{-\kappa_s x_3}). \tag{9}$$

Both τ_{13} and τ_{33} have contributions from both longitudinal and shear partial wave components decaying with different rates according to $e^{-\kappa_d x_3}$ and $e^{-\kappa_s x_3}$, respectively. For both stresses the amplitudes of the longitudinal and shear components must be equal so that the combined stress could be zero on the free surface. As a result, they must have identical functional dependence on x_3, i.e., in any plane parallel to the surface the ratio between the shear and normal stresses is constant.[6] What is more surprising that this ratio turns out to be again the same as the amplitude ratio between the vector and scalar potentials:

$$\frac{\tau_{13}}{\tau_{33}}=-\frac{i2k_R\kappa_d}{k_R^2+\kappa_s^2}=-i\zeta. \tag{10}$$

It should be mentioned here that a similar but more limited relationship exists for both symmetric and asymmetric Lamb waves propagating in a free plate.[18] The ratio between the normal and tangential surface displacements, which is this time also a function of frequency, is identical to the amplitude ratio between the vector and scalar potentials on the surface. However, in the case of a free plate, the stress ratio is depth dependent and distinctly different from the displacement ratio on the surface. It is interesting to consider whether a similar relationship exists in the case of the corresponding Rayleigh-type surface wave propagating on the free surface of an anisotropic half-space. Generally, three partial waves are required to satisfy the stress-free boundary conditions and the particle motion no longer lies in the sagittal plane. Since the stress ratios (τ_{23}/τ_{33} and τ_{13}/τ_{33}) in a given plane parallel to the surface are depth dependent, they cannot be related to the displacement ratios (u_2/u_3 and u_1/u_3) on the surface. It was also found that the aspect ratio of the elliptical particle trajectory produced by the dispersive Rayleigh-type surface wave on a curved surface is different from that of the nondispersive wave on a flat surface and does not possess a direct correlation to either the potential ratio or the stress ratio.[11] In other words, the above described uniform nature of the amplitude ratio appears to be limited to the classical Rayleigh wave propagating on the free surface of an isotropic elastic half-space only.

II. DISCUSSION AND CONCLUSION

Although the theory of Rayleigh wave propagation has been widely investigated and is very well understood, its behavior still holds some surprises. We showed that the ratio of the vector potential to the scalar potential is equal to the ratio of the normal to transverse displacement components on the surface. Since the vector and scalar potentials are essentially theoretical abstractions introduced to facilitate the analysis of the problem but do not directly correspond to any real physical quantity, this observation by itself is not particularly important, although it is rather unexpected if we take into consideration the differentiation process necessary to calculate the displacement field from its potentials and that both displacement components depend on both potentials. Much more important is the observation that the aspect ratio of the elliptical particle trajectory on the surface is identically the same as the ratio between the shear and normal stresses in any plane parallel to the surface at any depth, i.e., exactly the same amplitude ratio also shows up between two fundamental physical quantities.

Of course the identity of the amplitude ratios for displacements at the surface on one side and stresses at any depth on the other side, which apparently has not been pointed out in the otherwise very comprehensive literature on Rayleigh wave propagation, cannot be very well attributed to pure coincidence only. It is a direct result of the requirement that just below the surface the energy flow be parallel to the surface. The acoustic Poynting vector is defined as[19]

$$\mathbf{P}=-\frac{\mathbf{v}^*\tau}{2}, \tag{11}$$

where $\mathbf{v}=-i\omega\mathbf{u}$ is the complex particle velocity vector, τ is the complex stress tensor, and $*$ indicates complex conjugation. The normal component of the Poynting vector can be written as follows

$$P_3=\tfrac{1}{2}i\omega(u_1^*\tau_{13}+u_3^*\tau_{33}). \tag{12}$$

Since u_3 and τ_{33} are pure real while u_1 and τ_{13} are pure imaginary, P_3 is always pure imaginary, i.e., the average acoustic power transmission is always parallel to the surface. On the surface P_3 is identically zero since both τ_{13} and τ_{33} vanish. However, just under the surface even the oscillating part of the normal acoustic power, i.e., the imaginary part of P_3 must vanish therefore $\tau_{13}/\tau_{33}=-u_3^*/u_1^*$. As we mentioned above, u_3 is pure real, therefore $u_3^*=u_3$, but u_1 is pure imaginary, therefore $u_1^*=-u_1$. Consequently, $\tau_{13}/\tau_{33}=u_3/u_1$, as we have found by simple inspection of the displacement and stress fields. Further below the surface, the oscillating part of the normal acoustic power is not necessarily zero therefore the amplitude ratio u_3/u_1, which varies with depth, is not necessarily equal to the stress ratio τ_{13}/τ_{33}, which does not vary with depth.

The same argument also explains why the common amplitude ratio does not apply to either dispersive Rayleigh-type surface waves propagating on a cylindrical surface, nondispersive Rayleigh-type surface waves propagating on an anisotropic substrate in nonsymmetry directions, or to Lamb waves propagating in flat plates. The main reason is that the stress ratio in all these cases is depth dependent therefore we cannot relate the displacement ratio at the surface to the stress ratio somewhere else. In addition, on a concave cylindrical surface, the Rayleigh wave becomes leaky into the bulk of the solid so that the Poynting vector is definitely not parallel to the surface.

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