Diffraction Correction for Radiation Force Measurement on an Ideal Plane Reflector

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Summary
Discussion is given of a rigorous diffraction correction for acoustic radiometry by radiation-force measurement on a plane reflector. As an example, analytical results are presented for circular transducers of different aperture-to-wavelength ratio.

Zusammenfassung
In dieser Arbeit wird eine strenge Beugungskorrektur für die Messung des Strahlungsdrucks auf einen idealen ebenen Reflektor diskutiert. Als Beispiel werden analytische Ergebnisse für kreisförmige Wandler mit unterschiedlichem Verhältnis von Durchmesser zur Wellenlänge vorgestellt.

Sommaire
On exprime et discute une correction rigoureuse permettant de tenir compte des effets de diffraction sur les radiomètres acoustiques mesurant la pression de radiation s’exerçant sur un réflecteur plan. A titre d’exemple, on présente des résultats analytiques pour des transducteurs circulaires affectés de différents rapports ouverture/longueur d’onde.

Radiation force measurement is widely used to determine the total acoustic power conveyed by an ultrasonic beam. The sensitivity $s$ of such a system can be defined as the ratio between the measured radiation force $F_R$ and the total acoustic power $P_a$ to be determined. In practice, $s$ is somewhat uncertain due to its dependence on the beam divergence, therefore the so called nominal sensitivity $s_n$ is used to calculate the acoustic power on the assumption that the measured radiation force is produced by a plane wave impinging on the target. The measuring error due to this simplifying assumption can be taken into consideration by introducing the so called diffraction correction $e_d$:

$$e_d = \frac{s_n}{P_a} = \frac{s}{s_n}.$$  (1)

Langevin radiation force

We are to determine the Langevin radiation force produced by a laterally unconfined acoustic beam on a rigid plane target, i.e. the mean acoustic pressure along the reflector in Eulerian coordinates.

It is well known from the literature that the nominal sensitivity for plane wave irradiation at perpendicular incidence $s_n = 2/c$, where $c$ denotes the sound velocity in the surrounding medium. The actual sensitivity is much more difficult to determine, because the fine structure of the acoustic field must be taken into consideration. The Eulerian time average of the acoustic pressure $p_2$ can be determined from the complex amplitude of the velocity potential $\Phi_0$:

$$p_2 = \frac{\Phi_0}{4} (k^2 |\Phi_0|^2 - |\text{grad} \Phi_0|^2),$$  (2)

where $\Phi_0$ and $k$ denote the density and the wave number, respectively. Formula (2) expresses the fundamental theorem due to Langevin, quoted by P. Biquard [1], and later derived independently also by King [2]. In the presence of a reflector $\Phi_0$ can be written as the sum of the incident field $\Phi_i$ and the reflected one $\Phi_r$. Let us have an ideal infinite plane reflector of very high acoustic density at the $z = 0$ plane of a Cartesian system. The resultant velocity potential can be written as follows:

$$\Phi_0(x, y, z) = \Phi_i(x, y, z) + \Phi_r(x, y, z)$$

$$= \Phi_i(x, y, z) + \Phi_i(x, y, -z).$$  (3)
According to eq. (3), along the reflector \( z = 0 \)

\[
\Phi_0(x, y) = 2 \Phi_1(x, y),
\]

\[
\frac{\partial}{\partial x} \Phi_0(x, y) = 2 \frac{\partial}{\partial x} \Phi_1(x, y),
\]

\[
\frac{\partial}{\partial y} \Phi_0(x, y) = 2 \frac{\partial}{\partial y} \Phi_1(x, y),
\]

\[
\frac{\partial}{\partial z} \Phi_0(x, y) = 0.
\]

The mean acoustic pressure \( p_2 \) can be given directly as a function of the incident velocity potential amplitude \( \Phi_1 \):

\[
p_2(x, y) = q_0 \left( k^2 \left| \Phi_1(x, y) \right|^2 - \left| \frac{\partial}{\partial x} \Phi_1(x, y) \right|^2 \right).
\]

The total radiation force on the plane reflector can be determined by integrating \( p_2 \) over the whole \( z = 0 \) plane.

\[
f_R = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_2(x, y) \, dx \, dy
\]

\[
= q_0 \int_{-\infty}^{\infty} \left( k^2 \left| \Phi_1(x, y) \right|^2 - \left| \frac{\partial}{\partial x} \Phi_1(x, y) \right|^2 \right) \, dx \, dy.
\]

It seems to be very useful to introduce the following Fourier transforms:

\[
F\{\Phi_1(x, y)\} = F(k_x, k_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_1(x, y) \cdot e^{-i(k_x x + k_y y)} \, dx \, dy,
\]

\[
F\left\{ \frac{\partial}{\partial x} \Phi_1(x, y) \right\} = i k_x F(k_x, k_y),
\]

\[
F\left\{ \frac{\partial}{\partial y} \Phi_1(x, y) \right\} = i k_y F(k_x, k_y).
\]

According to the well known Parseval theorem, eq. (6) can be written into the more convenient form of

\[
f_R = q_0 \int_{-\infty}^{\infty} |F(k_x, k_y)|^2 (k^2 - k_x^2 - k_y^2) \, dk_x \, dk_y.
\]

In connection with eq. (8), we must emphasize that the velocity potential distribution is a special function which satisfies the so called wave equation, therefore \( F(k_x, k_y) = 0 \) if \( k_x^2 + k_y^2 > k^2 \). Furthermore, the Fourier transform of the velocity potential in an arbitrary \( z = z_0 \) plane can be written as

\[
F_{z_0}(k_x, k_y) = F(k_x, k_y) e^{i(k_x x_0 + k_y y_0 - z_0 k_z)},
\]

i.e. the modulus of the Fourier transform, which we shall call angular spectrum, is independent of the position of the reference plane if it remains perpendicular to the \( z \) axis [3]. According to this argument, eq. (8) yields the important result that the total radiation force on an infinite plane reflector due to an arbitrary acoustic beam is entirely independent of the distance between the transducer and the target if the relative direction of the beam with respect to the target remains the same.

Formulæ (6) and (8) offer two different alternatives of determining the total radiation force. We can substitute the velocity potential distribution along an arbitrary \( z = z_0 \) plane into eq. (6), or calculate the angular spectrum and substitute the result into eq. (8). However, it is extremely difficult to determine the velocity potential distribution along any plane other than one infinitely far from the transmitter of limited aperture, i.e. in the farfield. On the other hand, it is well known that the angular spectrum is very closely related to the modulus of the far-field distribution, therefore the second alternative also necessitates the determination of the far-field distribution.

**Far-field distribution**

The principal properties of the far-field distribution generated by a transmitter of limited aperture \( A \) can be derived by using Fig. 1. The complex amplitude of the surface velocity is \( v_0(x', y') \) inside the aperture and zero elsewhere. The elementary velocity potential component \( d\Phi_p \) at the \( P(x, y, z) \) reference point due to the \( dA(x', y') \) transmitter

![Fig. 1. Geometrical arrangement for calculation of the far-field distribution (left scale: 1; right scale 2).](attachment:image.png)
element can be given as [4]:

\[ \mathbf{d}\Phi = \frac{v_0(x', y')}{2\pi} \frac{e^{ikr'}}{r'} \, dA, \]  

(10)

where \( r' \) denotes the distance between \( dA \) and \( P \). The resultant velocity potential \( \Phi_P \) can be calculated by integrating \( d\Phi_P \) over the whole transmitter aperture.

\[ \Phi_P = \frac{1}{2\pi} \int_A v_0(x', y') \frac{e^{ikr'}}{r'} \, dx' \, dy'. \]

(11)

This general solution can be greatly simplified for points in the far-field:

\[ r' = \sqrt{(x-x')^2 + (y-y')^2 + z^2} \]

(12)

can be approximated by

\[ r' = r - \frac{x}{r} x' - \frac{y}{r} y', \]

(13)

where \( r = \sqrt{x^2+y^2+z^2} \) denotes the distance between \( P \) and the centre of the transmitter. Substituting eq. (13) into eq. (11), we get the far-field distribution of the velocity potential \( \Phi_t \):

\[ \Phi_t = \frac{e^{ikr}}{2\pi r} \int_A v_0(x', y') e^{-i(x'k_x + y'k_y)} \, dx' \, dy'. \]

(14)

In the denominator the second two terms of eq. (13) were neglected with respect to \( r \). The Fourier transform of the surface velocity distribution \( T(k_x, k_y) \) can be recognized easily in eq. (14).

\[ T(k_x, k_y) = \frac{1}{2\pi} \int_A v_0(x', y') e^{-i(x'k_x + y'k_y)} \, dx' \, dy', \]

(15)

where \( k_x \) and \( k_y \) are the spatial frequency components. From eq. (14), we can write the far-field distribution as a spherical wave weighted by the so-called directivity function \( D \).

\[ \Phi_t = \frac{e^{ikr}}{r} D \left( \frac{x}{r}, \frac{y}{r} \right), \]

(16)

where \( D \) includes also the complex amplitude:

\[ D \left( \frac{x}{r}, \frac{y}{r} \right) = T \left( k \frac{x}{r}, k \frac{y}{r} \right). \]

(17)

Modification of the Fraunhofer formula

According to the widely used Fraunhofer approximation, the far-field distribution can be found directly from the Fourier transform of the aperture distribution itself [5]:

\[ |\Phi_t(x, y, z)| \approx \frac{k}{z} \left| T \left( k \frac{x}{z}, k \frac{y}{z} \right) \right|. \]

(18)

Whenever the aperture is considerably larger than the wavelength, the angular spectrum can be approximated by the modulus of \( T(k_x, k_y) \), i.e.:

\[ z |\Phi_t(x, y, z)| \approx k |F(k_x, k_y)|, \]

(19a)

where

\[ k_x = k \frac{x}{\sqrt{k^2 + y^2 + z^2}} \]

(19b)

and

\[ k_y = k \frac{y}{\sqrt{k^2 + y^2 + z^2}}. \]

(19c)

In acoustics, the Fraunhofer approximation is usually not acceptable, because the aperture-to-wavelength ratio is too small, and \( x, y \) and \( k_x, k_y \) are not negligible with respect to \( r \) and \( k \). In the following, we shall prove that the more accurate

\[ \sqrt{x^2+y^2+z^2} |\Phi_t(x, y, z)| = \sqrt{k^2-k_x^2-k_y^2} |F(k_x, k_y)| \]

(20)

relationship should be used instead of the Fraunhofer approximation. According to the Parseval theorem, we can write for an arbitrary \( z = z_0 \) plane that

\[ \iint_{-\infty}^{\infty} |\Phi(x, y, z_0)|^2 \, dx \, dy = \iint_{-\infty}^{\infty} |F_{x}(k_x, k_y)|^2 \, dk_x \, dk_y. \]

(21)

In the far-field, we can choose an unlimited \( \Delta x \Delta y \) area which at the same time subtends an infinitesimal viewing angle from the transmitter, simply by increasing \( z_0 \). In the limiting case, the far-field distribution over this \( \Delta x \Delta y \) area approximates a plane wave with \( k_x = k x/r \) and \( k_y = k y/r \) spatial frequency components. Furthermore, the \( \Delta x k_x \) and \( \Delta y k_y \) products can be increased without limit, therefore only the \( x \times x + dx, y \times y + dy \) area contributes to the \( k_x \times k_x + dk_x, k_y \times k_y + dk_y \) spatial frequency components in \( F(k_x, k_y) \). Due to this filtering effect of the far-field distribution, in the limiting case \( z \to \infty \), eq. (21) simplifies to the equality of the integrals:

\[ |\Phi_t(x, y)|^2 \, dx \, dy = |F(k_x, k_y)|^2 \, dk_x \, dk_y. \]

(22)

The direct relationship between \( |\Phi_t(x, y)| \) and \( |F(k_x, k_y)| \) can be derived by expressing \( dk_x \, dk_y \) by the so-called Jacobi determinant \( D_j \):

\[ dk_x \, dk_y = dx \, dy \left| D_j \right|. \]

(23a)
where from eqs. (19 b) and (19 c)

$$D = \frac{\partial k_x}{\partial x} \frac{\partial k_y}{\partial y} - \frac{\partial k_x}{\partial y} \frac{\partial k_y}{\partial x} = \frac{z^2}{(x^2 + y^2 + z^2)^2}.$$  \quad (23 b)

By substituting eq. (23 b) into eq. (23 a), and in turn into eq. (22):

$$\Phi_t(x, y) = F(k_x, k_y) \left| \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right|.$$  \quad (24)

Eq. (24) can be rearranged into the more convenient form of eq. (20) by applying the

$$\frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{\sqrt{k^2 - k_x^2 - k_y^2}}{k}$$  \quad (25)

proportionality. Of course the resultant (20) relationship yields the Fraunhofer approximation for sufficiently large apertures and low spatial frequency components, i.e. when \(\sqrt{k^2 - k_x^2 - k_y^2} \approx k\) and \(\sqrt{x^2 + y^2 + z^2} \approx z\).

Let us substitute eq. (16) into (20):

$$f_k = \phi_0 \int_{k} \left| T(k_x, k_y) \right|^2 \frac{dk_x, dk_y}{k}.$$  \quad (26)

The final result can be written by using (8) and (17):

$$f_k = \phi_0 \int_{k} \left| T(k_x, k_y) \right|^2 \frac{dk_x, dk_y}{k},$$  \quad (27)

where \(K\) denotes the spatial frequency domain \(k^2 - k_x^2 - k_y^2 \approx 0\). According to our results, the total radiation force on an infinite plane reflector due to an acoustic beam radiated by a transmitter from an arbitrary distance can be directly determined from the directivity function of the transmitter. If the transmitter aperture is parallel with the mirror, i.e. the beam is aimed perpendicular at the target, the directivity function can be expressed simply by the Fourier transform of the surface-velocity distribution.

**Piston radiator**

We shall use the piston radiator as a model for an ultrasonic transmitter. Of course it is an approximation only, but a much better one than the plane wave approximation, and as for the crucial directivity function of the acoustic beam, it is a fairly good one. The surface velocity is constant \(v_0\) inside the circular aperture of radius \(a\), and zero elsewhere. For axisymmetric cases, the two-dimensional Fourier transform simplifies to the zero order Hankel transform:

$$T(k) = \int_{0}^{a} v_0(t) J_0(tk) \, dt.$$  \quad (28)

where \(r^2 = x^2 + y^2\) and \(k^2 = k_x^2 + k_y^2\), and \(J_0\) denotes the first kind Bessel function of zero order. In case of a piston radiator

$$T(k) = v_0 \int_{0}^{a} J_0(tk) \, dt = v_0 a^2 \frac{J_1(a k)}{a k}.$$  \quad (29)

Eq. (27) can be rewritten into an axisymmetric form as

$$f_R = \frac{\phi_0}{2\pi} \int_{0}^{\infty} \left| T(k) \right|^2 k \, dk,$$  \quad (30)

Finally, let us substitute eq. (29) into (30).

$$f_R = v_0 \int_{0}^{\infty} \frac{J_1^2(a k)}{a k} k \, dk.$$  \quad (31)

Since [6]

$$\int_{0}^{\infty} \frac{J_1^2(\xi)}{\xi} d\xi = \frac{\pi}{4} \int_{0}^{\infty} (1 - J_0(\xi') - J_1(\xi')) \, d\xi'$$  \quad (32)

we get the simple result that

$$f_R = v_0 \int_{0}^{a} \pi d_0 \left( 1 - J_0^2(a k) - J_1^2(a k) \right).$$  \quad (33)

The total acoustic power radiated by the vibrating piston [7]

$$P_a = \frac{1}{2} v_0 a^2 \phi_0 \left( 1 - J_1(2a k) \right),$$  \quad (34)

and the formerly defined sensitivity

$$s = \frac{f_R}{P_a} = \frac{2}{c} \frac{1 - J_0(a k) - J_1(a k)}{1 - J_1(2a k)/a k}.$$  \quad (35)

For \(a k \to \infty\) we get the well known nominal sensitivity \(2/c\). Finally, the diffraction correction can be calculated by (1)

$$e_a = \frac{1 - J_0^2(a k) - J_1^2(a k)}{1 - J_1(2a k)/a k}.$$  \quad (36)

For the sake of simplicity, let us regard \(a k\) as the scale of the aperture-to-wavelength ratio. The diffraction correction calculated by eq. (36) is shown in Fig. 2. For \(a k \to 0\), \(e_a = 0.5\), i.e. in the worst case of spherical irradiation the total radiation force will be only half of the radiation force due to plane wave irradiation of the same total acoustic power. The diffraction correction steeply increases with increasing aperture-to-wavelength ratio, therefore the radiation force measurement on an infinite plane reflector can be used to determine the total acoustic power conveyed by acoustic beams of relatively high divergence too.
Conclusions

In acoustic radiometry based on radiation force measurement, the unknown acoustic power is calculated from the measured radiation force by assuming that this force is due to a plane wave impinging on the target. The actual spatial distribution of the measured acoustic beam is usually neglected because (i) the radiation force measurement itself is rather inaccurate and (ii) although we may have certain information on the actual divergence of the acoustic beam to be examined, such as the aperture-to-wavelength ratio of the transducer, we do not know how to take its effect into consideration. The considerably increased accuracy of the sophisticated radiation force techniques makes it important that the nominal sensitivity of the system should be corrected according to the divergence of the acoustic beam. The first step in this direction is the development of a theoretical method to calculate the necessary diffraction corrections for precise acoustic radiometry.

It has been shown by the theoretical analysis, based on the Parseval theorem, that the total radiation force on an infinite plane reflector is independent of the distance between the transmitter and the target if the relative direction of the beam does not change. It was found that the angular spectrum of the acoustic beam is especially suitable for the calculation of the actual sensitivity, and we derived the exact relationship between the angular spectrum and the far-field distribution of the acoustic field. In this way, we arrived to the simple result that the total radiation force can be calculated directly from the directivity function of the acoustic beam. As a simple example, the diffraction correction was calculated for a piston radiator as a function of the aperture-to-wavelength ratio (the result is closely analogous to the so-called Lommel diffraction correction [8] for precise measurement of acoustic absorption).

The diffraction correction for typical ultrasonic transducers of $\alpha k \approx 100$ is about 0.5%, and is usually negligible. On the other hand, in all practical applications, the reflected beam should be absorbed in order to avoid disturbing interference. This means that the acoustic beam must not be directed perpendicularly to the plane reflector, or a different type, e.g. conical target should be used. In these cases, the diffraction correction may be considerably higher, because when determining the total radiation force, the higher angular frequency components are taken into account by a reduced weighting-factor proportional to the cosine of the angle of incidence on the target. At close to perpendicular incidence, the weighting factor decreases but very slowly with increasing angular frequency, while at smaller angle of incidence, the steepness of the cosine function is much greater, so the weighting factor decreases sharply. Hence a further analysis should be made to determine the diffraction corrections for other more practicable targets.

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References