Longitudinal guided wave propagation in a transversely isotropic rod immersed in fluid

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The propagation of longitudinal guided waves in fluid-loaded transversely isotropic rods has been investigated based on the superposition of partial waves. The numerical results indicate that fluid loading causes not only significant attenuation via energy leakage into the fluid but also strongly affects the phase velocity of the modes in certain frequency ranges. There is an apparent "mode switching" between the two lowest-order modes when the rod is loaded by a relatively high-density fluid. This phenomenon is analogous to the anomalous topology previously observed in the Lamb wave spectra of low-density water-loaded plates. © 1995 Acoustical Society of America.

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INTRODUCTION

Acoustic guided wave propagation along cylindrical rods is of great practical importance in numerous areas of acoustics. For example, such guided modes are often used to evaluate the material properties of thin metal wires, optical fibers, and reinforcement filaments used in epoxy, metal, and ceramic matrix composites. Most of the theoretical works in this area considered clad rods consisting of an isotropic core and an arbitrary number of isotropic coatings. A comprehensive review of these studies was published by Thurston.1 More recent efforts further developed this microscopically isotropic approach by including the effects of imperfect interfaces between the coaxial layers.2 At sufficiently low frequencies, such clad rods might appear strongly anisotropic with one axis of symmetry along the geometrical axis of the rod.

Axisymmetric guided wave propagation in anisotropic rods of transversely isotropic nature have been first studied by Chree3 and more recently by Morse4 and Mirkys.5 Much less attention was paid to the case of anisotropic rods immersed in fluid. One of the few efforts to study this problem was a recent paper by Dayal on the free vibrations of a fluid-loaded transversely isotropic rod based on uncoupling the radial and axial wave equations by introducing scalar and vector potentials.6 However, the potentials presented in Ref. 6 satisfy only the radial equation of motion [Eq. (4)] but not the axial one [Eq. (5)], which produces major anomalies in the final results. For example, the velocity of the lowest-order longitudinal mode should asymptotically approach the c_b=\sqrt{E_a/\rho} static limit at low frequencies, where \(E_a\) denotes the axial Young's modulus and \(\rho\) is the density. In the case of a transversely isotropic rod, Young's modulus along the axis of symmetry [001] can be written with four of the five independent elastic constants as \(E_a=c_{33}-2c_{13}^2/(c_{11}+c_{12})\). The same result was derived by Chree,3 Morse4 and Mirkys5 in their previously cited papers, which is very different from the explicit result given in Eq. (38) of Ref. 6.

In the following, we derive the dispersion relation of longitudinal guided waves in fluid-loaded transversely isotropic rods by using the technique of superposition of partial waves. First, we introduce Christoffel's equation in cylindrical coordinates to determine the velocities and polarization directions of the two axisymmetric modes that can propagate along the axis of symmetry in an infinite medium of transversely isotropic material. Second, we satisfy the free boundary conditions on the cylindrical surface of the unloaded rod by combining these two partial waves and derive the dispersion equation for the free rod. Finally, we modify the boundary conditions to account for the fluid loading on the vibration of an immersed rod and calculate the resulting velocity change and leaky attenuation of the guided modes.

I. ANALYTICAL TECHNIQUE

Let us consider a transversely isotropic material with the axis of symmetry oriented along the \(z\) axis of an \((r,\theta,z)\) cylindrical coordinate system. The constitutive equation can be written as follows (we are going to follow the notation of Ref. 6 as closely as possible):

\[
\sigma_{rr} = c_{11} \frac{\partial U_r}{\partial r} + c_{12} \frac{U_r}{r} + c_{13} \frac{\partial w}{\partial z},
\]

(1a)

\[
\sigma_{\theta\theta} = c_{12} \frac{\partial U_r}{\partial r} + c_{11} \frac{U_r}{r} + c_{13} \frac{\partial w}{\partial z},
\]

(1b)

\[
\sigma_{zz} = c_{13} \frac{1}{r} \frac{\partial (rU_r)}{\partial r} + c_{33} \frac{\partial w}{\partial z},
\]

(1c)

and

\[
\sigma_{rz} = c_{44} \left( \frac{\partial U_r}{\partial z} + \frac{\partial w}{\partial r} \right),
\]

(1d)

where \(U_r\) and \(w\) denote the radial and axial displacements, respectively. The radial and axial equations of motion can be written as

\[
\frac{1}{r} \frac{\partial (rU_r)}{\partial r} - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} - \rho \frac{\partial^2 U_r}{\partial t^2} = 0
\]

(2a)

and

\[
\frac{\partial (rU_r)}{\partial r} + \frac{\partial w}{\partial z} + \sigma_{rr} = 0
\]

(2b)
where
\[
d_{1,2} = (c_{11} \gamma_{1,2} + c_{13} k \xi_{1,2}) J_0(\gamma_{1,2} a)
- \frac{c_{11} c_{12}}{a} J_1(\gamma_{1,2} a)
\]  \tag{7b}

and
\[
e_{1,2} = c_{44} (ik - \xi_{1,2} \gamma_{1,2} J_1(\gamma_{1,2} a).
\]  \tag{7c}

The sought dispersion relation between \(\omega\) and \(k\) can be obtained by finding the zeros of the secular determinant in Eq. (7a).

These results are essentially the same as previously derived by Chreif,\(^7\) Morse,\(^4\) and Mirsky\(^5\) and can be easily extended to fluid-loaded rods as well. On the surface of an immersed rod, the normal stress is not zero but rather equals to the pressure in the fluid, i.e., \(\sigma_r(a) = -p(a)\). The fluid pressure can be readily calculated from the previously determined normal velocity of the solid,
\[
\hat{U}_r(a) = -i \omega [A_1 J_1(\gamma a) + A_2 J_1(\gamma a)],
\]  \tag{8}

and the well-known radiation impedance of an oscillating cylinder,\(^7\)
\[
Z_{\text{rad}} = \frac{p(a)}{\hat{U}_r(a)} = -i \rho_f \omega H_0(\gamma a).\]  \tag{9}

Here, \(\rho_f\) is the density of the fluid, \(\gamma_f = \sqrt{k_f^2 - \omega^2}\), \(k_f = \omega/c_f\), \(c_f\) is the sound velocity in the fluid, and \(H_0\) and \(H_1\) denote the zero- and first-order Hankel functions, respectively. The boundary conditions for the fluid-loaded rod can be written by modifying Eq. (7) as follows:
\[
\begin{bmatrix}
\sigma_{r1}(a) + p(a) \\
\sigma_{r2}(a)
\end{bmatrix}
= \begin{bmatrix} d_1 + f_1 & d_2 + f_2 \\ e_1 & e_2 \end{bmatrix} \begin{bmatrix} A_1 \\
A_2 \end{bmatrix}
= \begin{bmatrix} 0 \\
0 \end{bmatrix},
\]  \tag{10a}

where
\[
f_{1,2} = -\rho_f \omega^2 H_0(\gamma a) J_1(\gamma_{1,2} a).
\]  \tag{10b}

II. RESULTS AND DISCUSSION

The wave numbers of the longitudinal guided modes propagating in free and fluid-loaded transversely isotropic rods can be calculated at any frequency by numerically searching for the zeros of the secular determinants given by Eqs. (7) and (10), respectively. We shall present the dispersion relation in terms of the frequency-dependent phase velocity \(c_p(\omega) = \omega/\text{Re}[k]\) and attenuation coefficient \(\alpha(\omega) = \text{Im}[k]\). Like in most similar calculations, due care must be taken in choosing the roots corresponding to physical solutions of the wave equation. The radial wave numbers \(\gamma_1, \gamma_2,\) and \(\gamma_f\) are defined only through their squares and, mathematically, both positive and negative square roots satisfy the dispersion equation. In the case of a free rod, the secular determinant is an even function of both \(\gamma_1\) and \(\gamma_2\), therefore both positive and negative roots are allowable. In the case of a fluid-loaded rod, the secular determinant is an even function of \(\gamma_f\) but not of \(\gamma_1\) and \(\gamma_2\) anymore because of the correction term given in Eq. (10b). When calculating the wave number of the leaky guided waves along the fluid-
loaded rod, the easiest way to assure that only the physically allowable roots are taken is by assuring that the resulting attenuation be always positive.

As an example, Fig. 1 shows the calculated dispersion curves for the first eight longitudinal modes in a transversely isotropic glass/epoxy composite rod (the material properties were taken from Table I in Ref. 6). Solid lines show the familiar dispersion curves for the rod in vacuum and the dotted lines in water. The phase velocity was normalized to \( c_b = \sqrt{E_a/\rho} \), where \( E_a = c_{33} - 2c_{13}^2/(c_{11} + c_{12}) \). The normalized velocity of the lowest-order longitudinal mode always approaches unity at low frequency in spite of the fluid loading. The reason for this is that the vibration asymptotically approaches a purely axial motion and, without a normal surface vibration component, there is no coupling between the rod and the viscosity-free fluid.

Generally, the guided wave spectrum is only slightly perturbed by the presence of the fluid. For some modes, the dispersion curves of the two cases essentially overlap each other. The most interesting effect is the "switch over" from the 1st mode to the 2nd one around \( \alpha f \approx 1 \), that appears due to the coincidence of the 1st mode in the free rod and the 2nd mode in the immersed rod for \( \alpha f < 1 \). This anomalous behavior has been previously observed in the leaky Lamb wave spectra of both isotropic and anisotropic plates\(^8,9\) and can be also seen in the leaky guided wave spectra of thin shells.\(^10\)

For plates, the switching between the two lowest-order symmetric modes occurs if the density of the fluid approaches the density of the solid, therefore the loading is very strong. In our case, the density ratio is somewhat lower, \( \rho_f/\rho \approx 0.53 \), but, of course, a rod is much more affected by fluid loading than a plate of the same material. Some other modes get very close without actually crossing each other (e.g., the 2nd and 3rd ones at around \( \alpha f \approx 2.5 \)).

Figures 2 and 3 show the corresponding attenuation spectra in the immersed rod. In Ref. 6, the attenuation coefficient is plotted as a function of the normalized frequency \( \alpha f \). However, the attenuation coefficient is also an independent function of frequency (or radius). Therefore, we plotted the normalized attenuation, which is defined as the leaky attenuation over a length equal to the radius of the rod and depends only on the \( \alpha f \) product (another possibility would have been to define the normalized attenuation as the attenuation coefficient divided by frequency). Figure 2 shows the two branches produced by the above-mentioned anomalous topology. The "low-attenuation" mode of the fluid-loaded rod follows the 1st true mode of the free rod at low frequencies, and the normalized attenuation is proportional to the cube of the normalized frequency, \( \alpha A (\alpha f)^3 \). This leaky mode approaches the 2nd true mode of the free rod at high frequencies, and the attenuation is much higher. As in the case of leaky Lamb wave propagation in an immersed plate, the "high-attenuation" mode follows the complex branch of the 2nd mode of the free rod and the attenuation is extremely high. This leaky mode approaches the 1st true mode of the free rod at high frequencies, and the normalized attenuation becomes proportional to the normalized frequency, i.e., \( \alpha A \propto \alpha f \). This behavior could be expected since the mode asymptoti-...
cally approaches the leaky Rayleigh mode as the wavelength becomes much smaller than the radius of the rod. It is well known that the velocity of the leaky Rayleigh mode is slightly higher than the true Rayleigh velocity and its attenuation is proportional to frequency. Both effects are determined by the density ratio between the fluid and the solid and are independent of the radius of the rod.

Figure 3 shows the normalized attenuation of four higher-order modes. The attenuation of the 6th and 8th modes are very small in this frequency range and they were omitted to reduce the confusion. The modes can be classified into two basic groups based on their behavior in the free rod at their cutoff frequencies, i.e., when their phase velocity is infinitely high and there is no energy propagation along the rod. For \( k = 0 \), the two eigenvalues of Christoffel’s equation [Eq. (4)] are \( \gamma_1 = \omega/c_{st} \), and \( \gamma_2 = \omega/c_{dr} \), where \( c_{st} = \sqrt{c_{44}/\rho} \) and \( c_{dr} = \sqrt{c_{11}/\rho} \) are the velocities of the pure shear and pure longitudinal waves propagating in the plane of isotropy, i.e., in the radial direction. At the cutoff frequency, those modes which are pure shear resonances produce only transverse vibrations at the surface therefore they are not attenuated by leakage into the fluid. The 3rd and 4th modes (as well as the omitted 6th and 8th modes) belong to this low-attenuation family, which we shall call class A. The other modes which are pure longitudinal resonances produce only normal vibrations at the surface therefore they are very strongly attenuated by leakage into the fluid. The 5th and 7th modes belong to this high-attenuation family, which we shall call class B. In the case of fluid-loaded rods, the phase velocity of class B modes reaches a maximum around the cutoff frequency of the corresponding mode in the free rod (which is above the velocity range displayed in Fig. 3). Below the cutoff frequency, these modes continue in very highly attenuated complex branches and their phase velocity drops to zero as the frequency further decreases. In the case of plates, the modes are usually classified into asymmetric and symmetric families. Based on their behavior at the cutoff frequency, class A modes in a rod correspond to asymmetric modes in plates and class B modes to symmetric ones. Of course, both class A and class B modes are axisymmetric vibrations of the rod therefore we cannot adapt the asymmetric/symmetric terminology used for plates. Finally, we should note that a normalized attenuation of 1 dB or higher renders the mode non-propagatory for all practical purposes. For example, even when the length of the rod is as small as ten times its diameter, i.e., it is barely a short stub, the corresponding leaky attenuation reaches 20 dB.

III. CONCLUSIONS

The propagation of longitudinal guided waves in fluid-loaded transversely isotropic rods has been investigated. The same problem has been recently investigated in Ref. 6, however the solution presented in that paper does not satisfy the wave equation. Because of this conceptual problem, the analytical results of Ref. 6 are only quasi-isotropic approximations for anisotropic materials, at best. Even for an isotropic rod, some of the predictions for the fluid-loaded case seem to be incorrect in that paper. The author claims that near the cutoff frequency the effect of water loading is minimal. This is not true for class B modes which are pure longitudinal resonances and therefore strongly attenuated at the cutoff frequency. Furthermore, the velocity of the leaky Rayleigh wave appears to be lower than that of the true Rayleigh mode, which is not possible.

We presented an alternative solution for transversely isotropic rods based on the superposition of partial waves. These results are exact, albeit numerical, solutions of the wave equation in homogeneous anisotropic rods of circular cross section. The calculated results exhibit the right asymptotic behavior both at low and high frequencies as well as the cutoff frequencies of individual modes. Our numerical results indicate that there is a “mode switching” between the two lowest-order modes when the rod is loaded by a relatively high density fluid. This phenomenon is analogous to the anomalous topology previously observed in the Lamb wave spectra of low-density water-loaded plates.

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