LETTERS TO THE EDITOR

This Letters section is for publishing (a) brief acoustical research or applied acoustical reports, (b) comments on articles or letters previously published in this Journal, and (c) a reply by the article author to criticism by the Letter author in (b). Extensive reports should be submitted as articles, not in a letter series. Letters are peer-reviewed on the same basis as articles, but usually require less review time before acceptance. Letters cannot exceed four printed pages (approximately 3000–4000 words) including figures, tables, references, and a required abstract of about 100 words.

Pulse propagation in an elastic medium with quadratic nonlinearity (L)

Jianmin Qu
Department of Civil and Environmental Engineering, Department of Mechanical Engineering, Northwestern University, Evanston, Illinois 60208

Peter B. Nagy
School of Aerospace Systems, University of Cincinnati, Cincinnati, Ohio 45221

Laurence J. Jacobs
College of Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332

(Received 25 July 2011; revised 16 December 2011; accepted 28 December 2011)

This letter examines the propagation of an acoustic pulse in an elastic medium with weak quadratic nonlinearity. Both a displacement pulse and a stress pulse of arbitrary shapes are used to generate the wave motion in the solid. By obtaining the explicit solutions for arbitrary pulse shapes, it is shown that for a sinusoidal tone-burst, in addition to a second order harmonic field, a radiation induced static strain field is also generated. These results help clarify some confusion in the recent literature regarding the shape of the propagating static displacement pulse.

© 2012 Acoustical Society of America. [DOI: 10.1121/1.3681922]

PACS number(s): 43.25.Dc, 43.25.Ed, 43.25.Qp, 43.25.Ba [ANN]

Pages: 1827–1830

This letter examines the propagation of an acoustic pulse in an elastic medium with weak quadratic nonlinearity. To begin, consider a half-space defined by $x \geq 0$, where $x$ is the Lagrangian (or material) coordinate describing the location of the material particle in the initial ($t = 0$) state. At any given time $t$, the displacement of the particle $x$ from its initial position is denoted by $u(x,t)$. Deformation of the elastic body can then be described by the Lagrangian strain

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2.$$  \hspace{1cm} (1)

We assume that the half-space is made of an elastic solid with quadratic nonlinearity, i.e., the normal (first Piola–Kirchhoff) stress is related to the Lagrangian strain/displacement gradient in the $x$-direction through

$$\sigma = \rho c^2 \left[ \varepsilon - \frac{\beta}{2} \frac{1}{\varepsilon^2} \right] = \rho c^2 \left[ \frac{\partial u}{\partial x} - \frac{\beta}{2} \left( \frac{\partial u}{\partial x} \right)^2 \right].$$  \hspace{1cm} (2)

where $\rho$ is the mass density, $c$ is the longitudinal phase velocity, and $\beta$ is the acoustic nonlinearity parameter, all for the elastic solid in the undeformed (initial) state.

The displacement equation of motion governing the wave propagation in the $x$-direction is

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\frac{\beta}{2} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2}. $$  \hspace{1cm} (3)

By a standard perturbation procedure, one may write the solution to Eq. (3) as

$$u(x,t) = u_1(x,t) + u_2(x,t), $$  \hspace{1cm} (4)

where $|u_1(x,t)| \gg |u_2(x,t)|$, or $u_2 = O(u_1^2)$, and

$$\frac{1}{c^2} \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} = 0, \quad \frac{1}{c^2} \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} = -\frac{\beta}{2} \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2}. $$  \hspace{1cm} (5)

The solution to the first expression of Eq. (5) that represents a forward propagating wave can be written as

$$u_1(x,t) = f(t - x/c). $$  \hspace{1cm} (6)

It then follows that the second expression of Eq. (5) can be written as

$$\frac{1}{c^2} \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} = g(t - x/c), $$  \hspace{1cm} (7)
where
\[ g(s) = \frac{\beta}{c^d} f'(s)f''(s), \tag{8} \]
and the prime denotes the derivative with respect to the argument of the function. By a direct substitution, one can show that the solution to Eq. (7) is given by
\[ u_2(x,t) = \frac{\beta x}{2c^d} \int_0^{t-x/c} f'(s)f''(s)ds + Dx + B(t-x/c), \tag{9} \]
where \( B(y) \) is an arbitrary function of \( y \) and \( D \) is an integration constant, both need to be determined by the boundary conditions and/or the consistency condition 1,2
\[ \frac{\partial u}{\partial t} = \frac{2c}{3\beta} \left( 1 - \beta \frac{\partial u}{\partial x} \right)^{3/2} - 1. \tag{10} \]

If \( f(s) \) is a smooth function for \( s \in (0,t-x/c) \), the integral in Eq. (9) can be carried out,
\[ u_2(x,t) = \frac{\beta x}{4c^d} \left( [f'(t-x/c)]^2 - [f'(0^+)]^2 \right) + Dx + B(t-x/c). \tag{11} \]
This is the general solution to the second order governing Eq. (5).

Now, we determine \( B(y) \) and \( D \) under different boundary conditions. First, consider the case where the displacement is prescribed on the boundary, i.e.,
\[ u(0, t) = u_0(t). \tag{12} \]
Consequently,
\[ u(0, t) = u_0(t), \quad u_2(0, t) = 0. \tag{13} \]
It is then easy to show that
\[ u_1(x,t) = f(t-x/c) = u_0(t-x/c). \tag{14} \]
The second order solution thus follows from Eq. (9) that
\[ u_2 = \frac{\beta x}{4c^d} \left( [f'(t-x/c)]^2 - [f'(0^+)]^2 \right), \tag{15} \]
where we have chosen \( B(t) = 0 \) in order to satisfy the boundary condition (13), and \( D = \beta [f'(0^+)]^2/(4c^2) \) to satisfy the consistency condition (10). Combining Eqs. (14) and (16) gives the solution under displacement boundary condition (12)
\[ u_D(x,t) = u_0(t-x/c) + \frac{\beta x}{4c^d} \left( [f'(t-x/c)]^2 - [f'(0^+)]^2 \right), \tag{16} \]
where the subscript \( D \) is to indicate that the solution is for the displacement prescribed boundary condition. Equation (16) was derived by Lamb 3 using a different method.

Next, consider the case where a traction is prescribed on the boundary,
\[ \sigma(0, t) = \sigma_0(t). \tag{17} \]
Making use of the constitutive law (2), one may expand the stress into
\[ \sigma(x,t) = \sigma_1(x,t) + \sigma_2(x,t), \tag{18} \]
where \( |\sigma_1(x,t)| \gg |\sigma_2(x,t)| \) and
\[ \sigma_1(x,t) = \rho c^2 \frac{\partial u_1}{\partial x}, \quad \sigma_2(x,t) = \rho c^2 \left( \frac{\partial u_2}{\partial x} - \frac{\beta}{2} \left( \frac{\partial u_1}{\partial x} \right)^2 \right). \tag{19} \]
The corresponding boundary conditions for \( \sigma_1(x,t) \) and \( \sigma_2(x,t) \) follow directly from Eqs. (17) and (18),
\[ \sigma_1(0, t) = \sigma_0(t), \quad \sigma_2(x,t) = 0. \tag{20} \]
Substituting Eq. (19) into Eq. (20) leads to
\[ \frac{\partial u_1(x,t)}{\partial x} \bigg|_{x=0} = \frac{\sigma_0(t)}{\rho c^2}, \quad \frac{\partial u_2}{\partial x} \bigg|_{x=0} = \frac{\beta}{2} \left( \frac{\partial u_1}{\partial x} \right)^2 \bigg|_{x=0} = \frac{\beta}{2} \left( \frac{\sigma_0(t)}{\rho c^2} \right)^2. \tag{21} \]
In this case, it is straightforward to show that
\[ u_1(x,t) = f(t-x/c) = \frac{1}{\rho c} \int_0^{t-x/c} \sigma_0(s)ds. \tag{22} \]
Substituting Eq. (22) into Eq. (11) in conjunction with the second expression of Eq. (21) leads to
\[ D = \frac{\beta}{4\rho^2 c^4} [\sigma_0(0^+)]^2, \quad B'(t) = -\frac{\beta [\sigma_0(t)]^2}{4\rho^2 c^3}. \tag{23} \]
Integrating the second expression of Eq. (23) yields
\[ B(t) = -\frac{\beta}{4\rho^2 c^3} \int_{0^+}^{t} [\sigma_0(s)]^2 ds, \tag{24} \]
where we had ignored the integration constant, since a constant in the displacement is irrelevant for traction-prescribed problems.

Finally, combining Eqs. (22)–(24) and (11) gives the solution under the traction-prescribed boundary condition
\[ u_T(x,t) = \frac{1}{\rho c} \int_{0^+}^{t-x/c} \sigma_0(s)ds + \frac{\beta x}{4\rho^2 c^3} [\sigma_0(t-x/c)]^2 \]
\[ - \frac{\beta}{4\rho^2 c^3} \int_{0^+}^{t-x/c} [\sigma_0(s)]^2 ds. \tag{25} \]
To elucidate some physical features of the solution obtained above, consider the propagation of a sinusoidal pulse of angular frequency \( \omega \). For convenience, define a rectangular pulse

\[
P(t) = H(t)H(t-t), \tag{26}
\]

where \( H(t) \) is the Heaviside step function, and \( \tau = 2\pi/\omega \) with \( n \) being a positive integer.

Now, let us begin with the displacement-prescribed boundary condition

\[
u_0(t) = U P(t) \sin \omega t. \tag{27}\]

Substituting Eq. (27) into Eq. (16) gives

\[
u_D(x,t) = \left[ U \sin (t-x/c) + \frac{\beta U^2 \omega^2 x}{8c^2} \cos 2\omega \left( t - \frac{x}{c} \right) \right. + \left. \frac{\beta U^2 \omega^2 x}{8c^2} \right] P(t-x/c). \tag{28}\]

We note that the first term on the right hand side of Eq. (28) is the original propagating pulse of frequency \( \omega \). The second term represents a propagating pulse of frequency \( 2\omega \) with linearly growing amplitude. The third term, \( \left( \beta U^2 \omega^2 x / 8c^2 \right) P(t-x/c) \) is the static portion of the displacement. It represents a propagating static pulse in that (1) at any fixed location \( x \), the displacement is a rectangular pulse in the time domain, thus the term pulse, (2) at any fixed time \( t \), the medium between \( x = c(t-\tau) \) and \( x = ct \) under goes a positive uniform strain, i.e., the displacement increases linearly from \( x = c(t-\tau) \) to \( x = ct \), thus the term “static,” and (3) this region of uniform strain moves in the positive \( x \)-direction with velocity \( c \), thus the term propagating.

Further, we note that the amplitude of the static displacement at a given point is proportional to the distance between this point and the boundary, and the proportional constant is \( \beta U^2 \omega^2 / 8c^2 \). If the signal for the static portion of the displacement is recorded by a receiver at location \( x_0 \), the recorded signal plotted as a function of time will be a “flat topped” rectangle of height \( (\beta U^2 \omega^2 / 8c^2)x_0 \). The length of the rectangle will be \( \tau \). This is consistent with the experimental observations of Refs. 4 and 5 and the numerical analysis based on the finite difference method.\(^6\) We note that the numerical analysis in Ref. 6 is indeed for displacement-prescribed boundary condition.

Also, for the time-harmonic case where \( \tau \to \infty \), one may reduce Eq. (28) to the time-harmonic solution obtained in Ref. 1,

\[
u_D(x,t) = \left[ U \sin \left( t - \frac{x}{c} \right) + \frac{\beta U^2 \omega^2 x}{8c^2} \cos 2\omega \left( t - \frac{x}{c} \right) \right. \left. + \frac{\beta U^2 \omega^2 x}{8c^2} \right] H \left( t - \frac{x}{c} \right). \tag{29}\]

Next, consider the traction-prescribed boundary condition

\[
\sigma_0(t) = -\rho c \omega U P(t) \cos \omega t. \tag{30}\]

Substituting Eq. (30) into (25) yields

\[
u_T(x,t) = U \sin \left[ \omega \left( t - \frac{x}{c} \right) \right] P(t-x/c) - \left. \frac{\beta U^2 \omega^2 x}{8c^2} \right) (2x-ct)P(t-x/c) + \frac{\beta U^2 \omega^2 x}{16c} \left[ \frac{2\omega x}{c} \cos 2\omega(t-x/c) - \sin 2\omega(t-x/c) \right] P(t-x/c). \tag{31}\]

Clearly, the second term on the right hand side of Eq. (31) represents the static displacement. As in the case of displacement-prescribed boundary condition, the static displacement is also a propagating pulse with its amplitude growing with propagation distance. However, if the signal for the static portion of the displacement is recorded by a receiver at location \( x_0 \), the recorded signal plotted as a function of time will not be “flat topped.” Instead, the top of the pulse will linearly decrease from \( (\beta U^2 \omega^2 / 8c^2)x_0 \) at the front edge to \( (\beta U^2 \omega^2 / 8c^2)(x_0 - \tau c) \) at the trailing edge of the pulse. In other words, if the signal for the static portion of the displacement is recorded by a receiver at location \( x_0 \), the recorded signal plotted as a function of time will not be a “flat topped” rectangle. Instead, it will be a “slant topped” trapezoid. Our results are in sharp contradiction with the original predictions of Yost and Cantrell\(^7\) who correctly predicted that the static strain is a flat topped pulse of magnitude \( \beta U^2 \omega^2 / (8c^2) \), but then incorrectly suggested that the static displacement is a right-angle triangle with a peak value of \( \beta U^2 \omega^2 / (8c^2)L \), where \( L = ct \) is the spatial length of the pulse. In fact, causality would dictate that the pulse shape cannot be a right-angle triangle predicted in Refs. 7 and 8. Information about the length of the pulse generated at \( x = 0, t = \tau \) cannot propagate faster than the velocity of sound, therefore it cannot influence the initial peak value of the quasi-static pulse. Further, the amplitude of the static pulse must increase with propagation distance for a given pulse length since the nonlinear propagating part of the effect is cumulative.

J.Q. and L.J.J. acknowledge the financial support by the National Science Foundation (Grant No. CMMI-0653883) and by the Air Force Office of Scientific Research (Grant No. FA9550-08-1-0241).